

General Lecture in Magister Teknik Sipil  
Universitas Diponegoro, 1 Feb 2014

# Isoparametric Finite Elements

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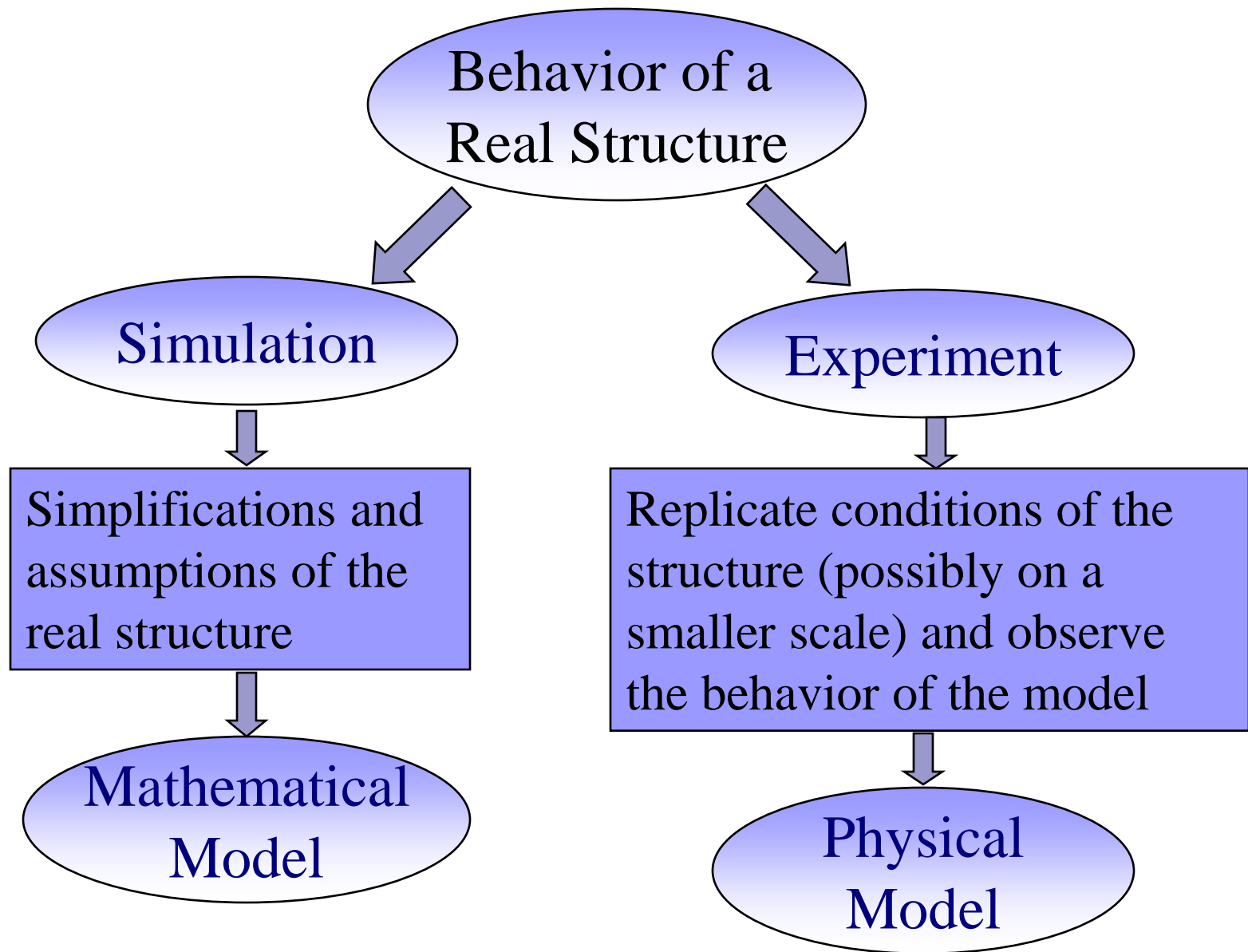
Petra Christian University  
Surabaya





# Lecture Outline

1. Overview of the FEM
2. Governing equations of plane-strain/plane-stress problems
3. Finite element formulation
4. Isoparametric elements
5. Element tests and applications
6. References



# An example of the FEM applications

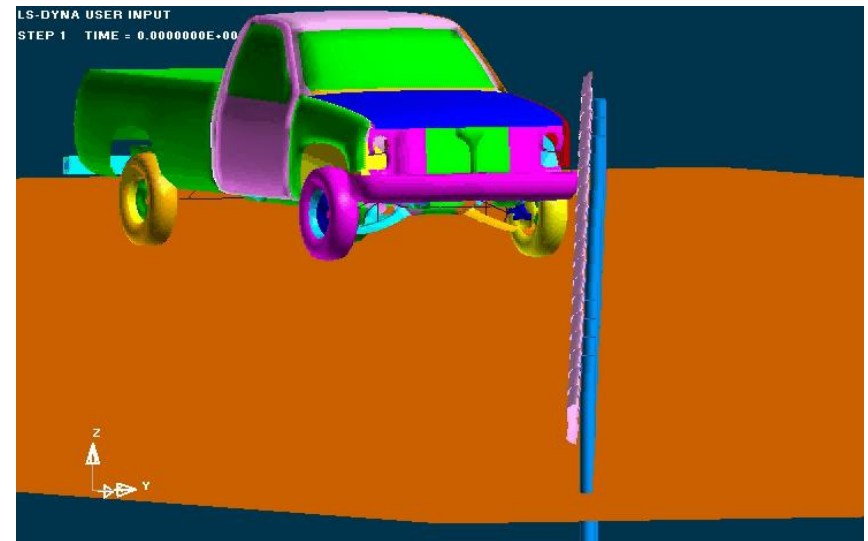
## Real experiment

It is often expensive or dangerous



## FE simulation

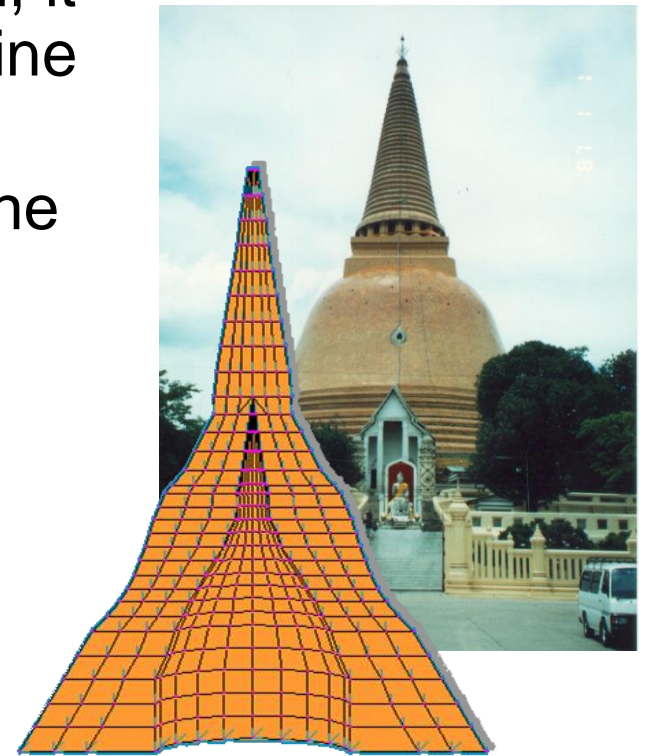
It replicates conditions of the real experiment



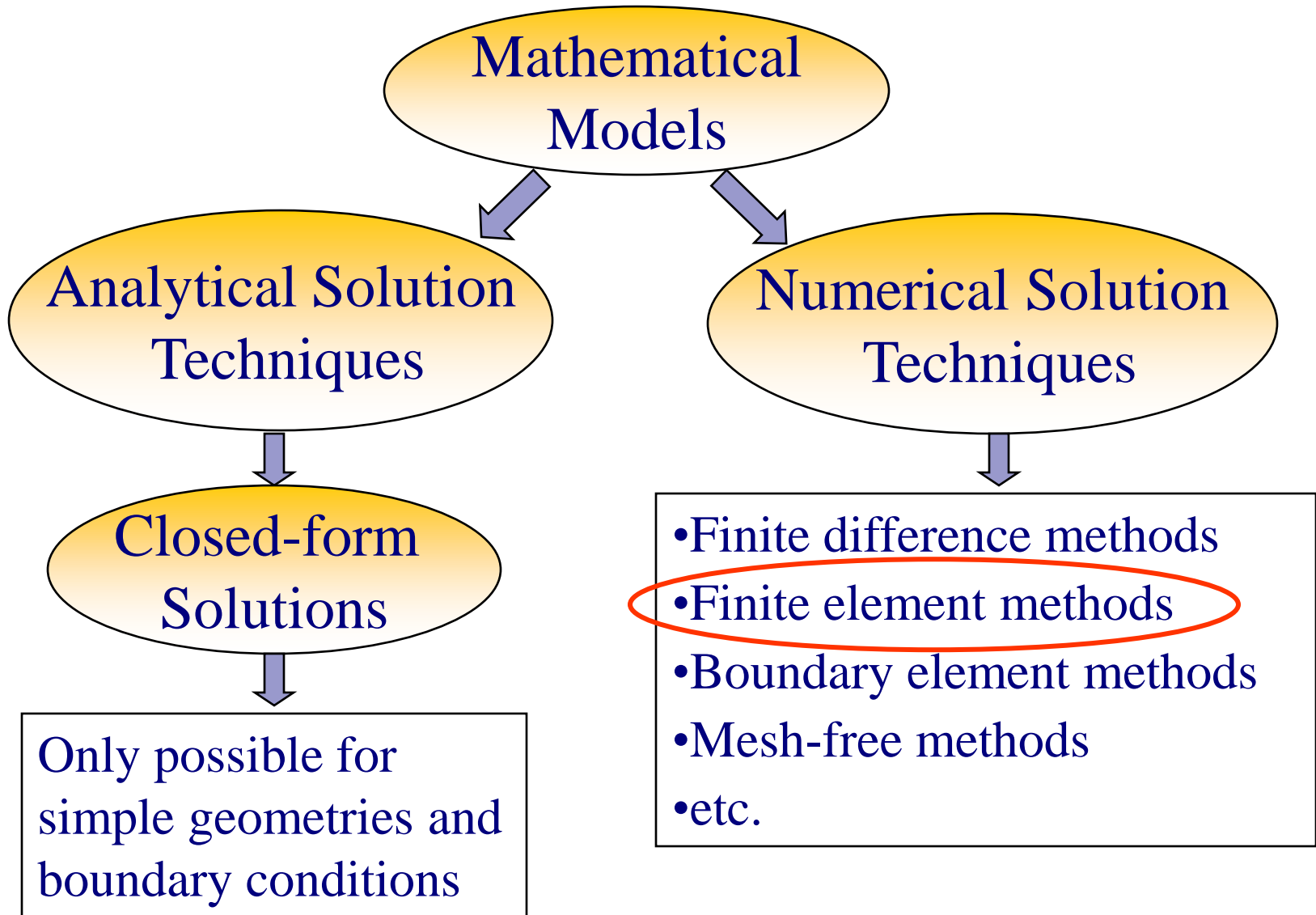
Source: W.J. Barry (2003), "FEM Lecture Slides", AIT Thailand

# The need for modeling

- A real structure cannot be analyzed, it can only be “load tested” to determine the responses
- We can only analyze a “model” of the structure (perform simulation)
- We need to model the structure as close as possible to represent the behavior of the real structure



Source: W. Kanok-Nukulchai

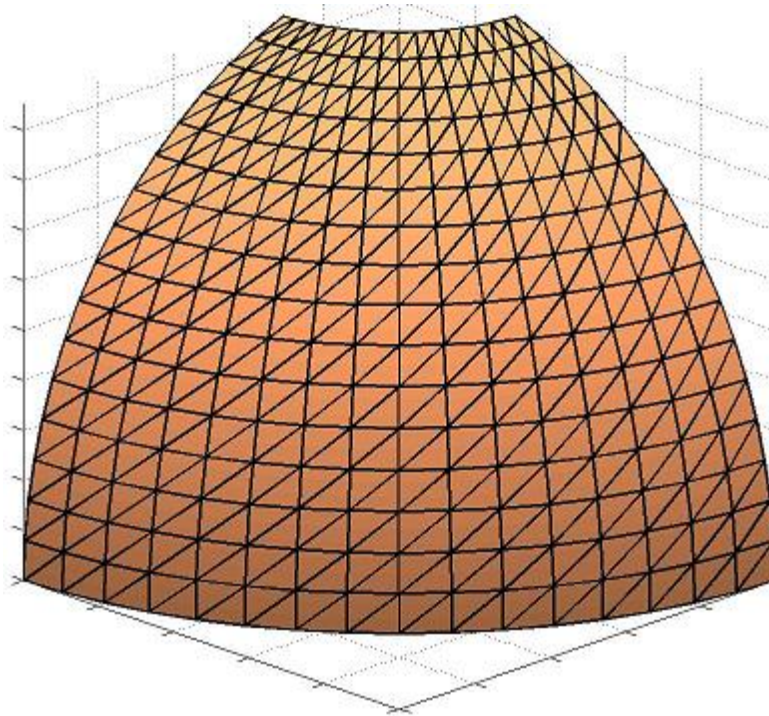


# What is FEM?

- It is a computational technique used to obtain **approximate solutions** of engineering problems.
  - The results are generally **not exact**.
  - However, the accuracy of the results can be improved either using finer mesh (***h*-refinement**) or higher degree elements (***p*-refinement**)

# Solution refinements in FEM

*h*-refinement



$h=1$



$h=1/2$

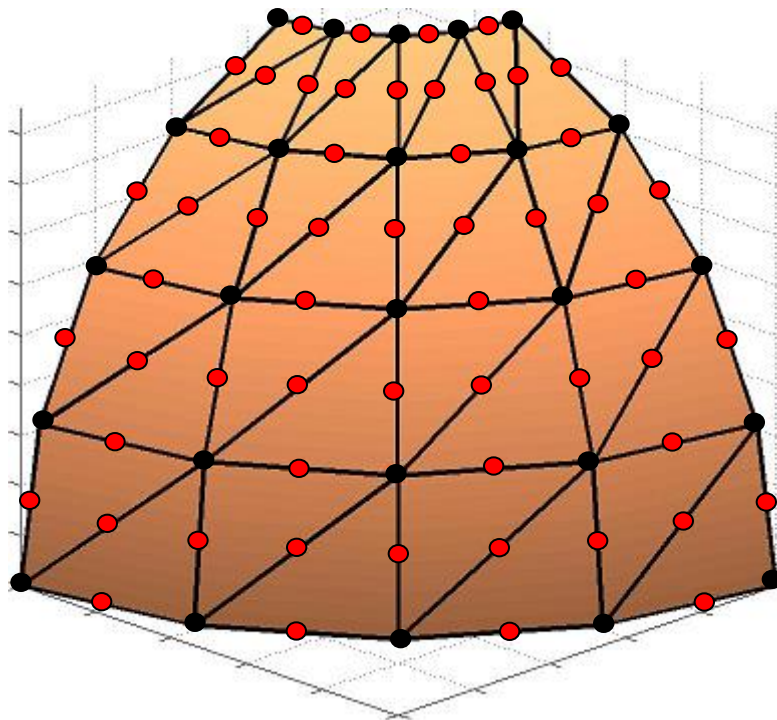


$h=1/4$



# Solution refinements in FEM (cont'd)

*p*-refinement



$$u = a + bx + cy$$



$$u = a + bx + cy + dx^2 + exy + fy^2$$

# Examples of FEM software

- For General purposes:  
NASTRAN, ANSYS, ADINA, ABAQUS, etc.
- For structural analysis, particularly in Civil Engineering:  
SANS, SAP, STAAD, GT STRUDL, MIDAS, DIANA, **STRAND 7**, etc.
- For building structures:  
ETABS, BATS etc.

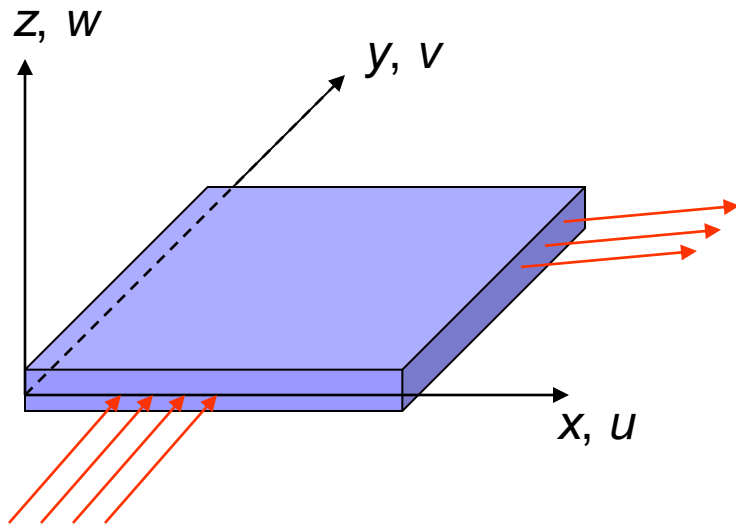
# Focus of lecture

- FEM is originated as a method of **structural analysis** but is now widely used in various disciplines such as heat transfer, fluid flow, seepage, electricity and magnetism, and others.
- The present discussion will focus on FEM for **structural analysis**, with the scope:
  - Plane stress/plane strain problems
  - Linear static analysis
  - Isoparametric formulation
    - Bilinear isoparametric quadrilateral element (Q4)

# Lecture Outline

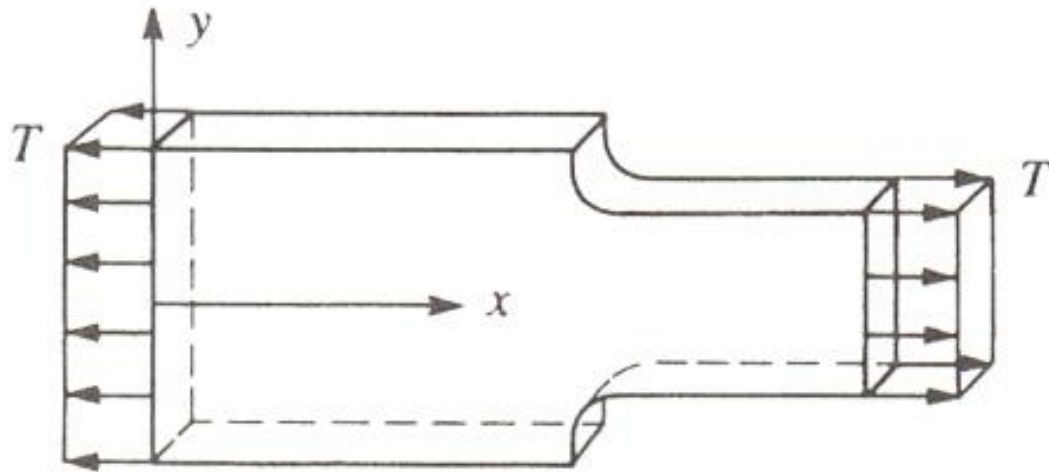
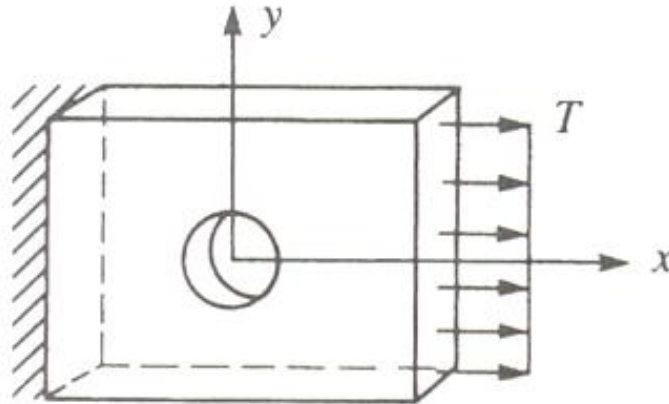
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# Plane stress

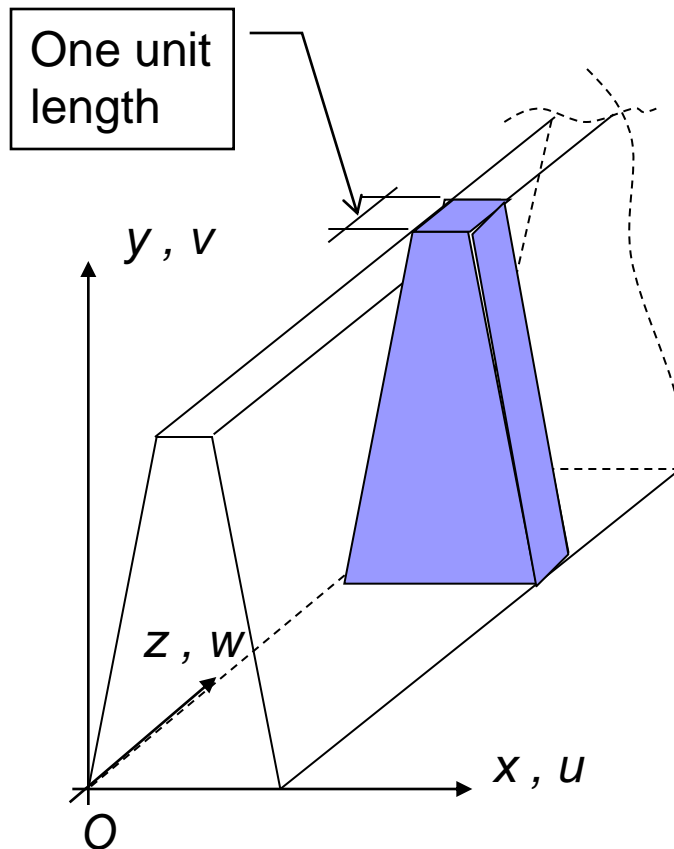


- A stress condition that prevails in a thin plate loaded only in its own plane, say  $xy$  plane, and without restraint in its perpendicular direction.
- $\sigma_z = \tau_{yz} = \tau_{zx} = 0$
- Typical examples are thin plates loaded in the plane of the plate.

# Example:

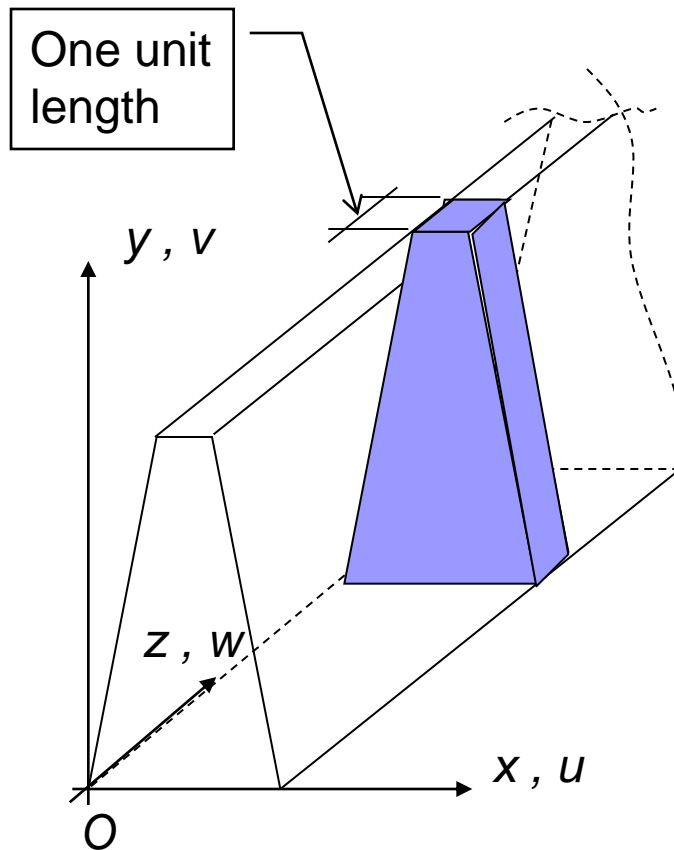


# Plane strain



- A deformation state in which
  - $w=0$  everywhere
  - $u=u(x, y)$
  - $v=v(x, y)$
- Thus,
  - $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$
- A state of strain in which the strain normal to the  $x$ - $y$  plane and the shearing strains  $\gamma_{xz}$  and  $\gamma_{yz}$  are zero.

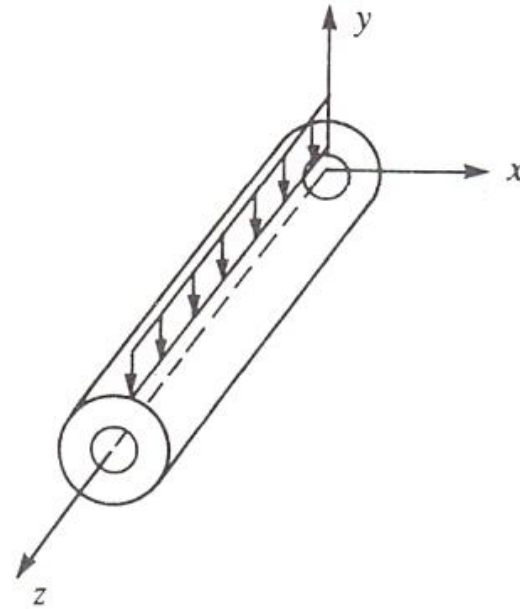
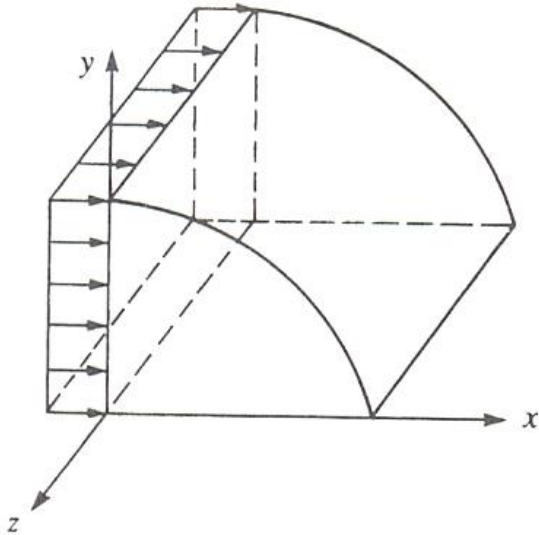
# Plane strain (cont'd)



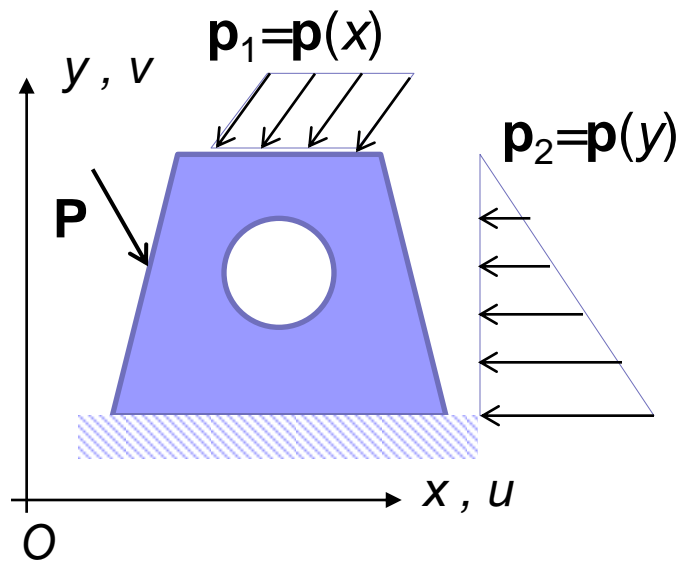
- The plane strain model is realistic for a long body with constant cross section subjected to planar loads that do not vary along the body.
- Examples:
  - A slice of an underground tunnel that lies along  $z$  axis
  - A slice of an earth retaining wall
- Only a unit thickness of the body is considered in an analysis using the plane strain model.



# Example:

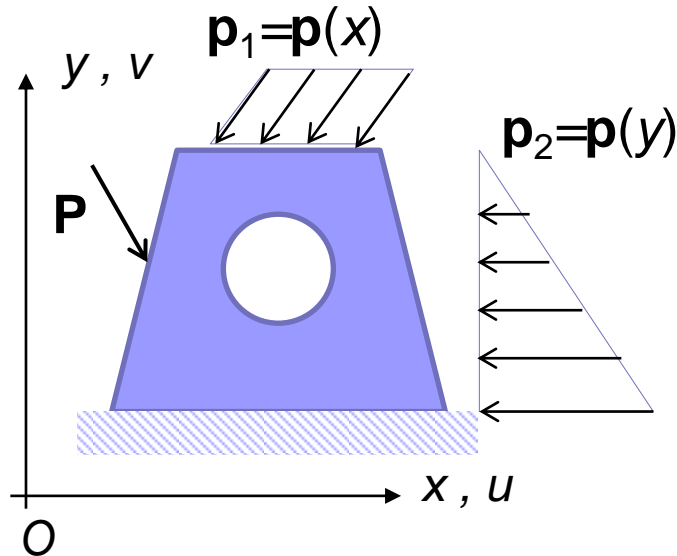


# Basic concepts from the theory of elasticity



Note that a force is a vector, while temperature is a scalar.

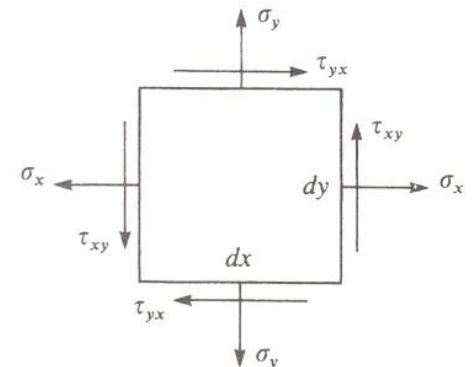
- Consider a plane stress/strain model of a body (structure) as illustrated here.
- The body is subjected to:
  - Concentrated force  $\mathbf{P}$
  - Distributed surface forces  $\mathbf{p}_1$  and  $\mathbf{p}_2$ , can be given in the unit of [force]/[area] or [force]/[length]
  - Body force  $\mathbf{b} = \mathbf{b}(x, y)$ , e.g. due to the self-weight of the body, [force]/[volume]
  - Temperature change  $T^{\circ}\text{C}$



- The **plane stress/strain problem** is: given the external loads, temperature change, and displacement boundary condition, find the displacement field, the strain field and the stress field.

# Output of an analysis

- The results of an analysis are:
  - Two displacement components, i.e.  $u$  and  $v$
  - Three strain components, i.e.  $\epsilon_x$ ,  $\epsilon_y$ ,  $\gamma_{xy}$
  - Three stress components, i.e.  $\sigma_x$ ,  $\sigma_y$ ,  $\tau_{xy}$
- In most FE softwares we can express the stress results in terms of:
  - Principal stresses
  - von Mises stress



# Governing equations

- Three basic set of equations in the theory of elasticity are:
  - Strain-displacement equations
  - Stress-strain equations
  - Equations of equilibrium

# Strain-displacement relations

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \text{atau} \quad \boldsymbol{\varepsilon} = \boldsymbol{D} \boldsymbol{u}$$

# Stress-strain relations

## ■ Plane stress

- For a body made up from isotropic materials, the stress-strain relation is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \frac{E\alpha T}{1-\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

- Notice that the **strain** in z direction may not be zero

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) + \alpha T$$

# Stress-strain relations (cont'd)

## ■ Plane strain

- For a body made up from isotropic materials, the stress-strain relation is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \frac{E\alpha T}{1-2\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

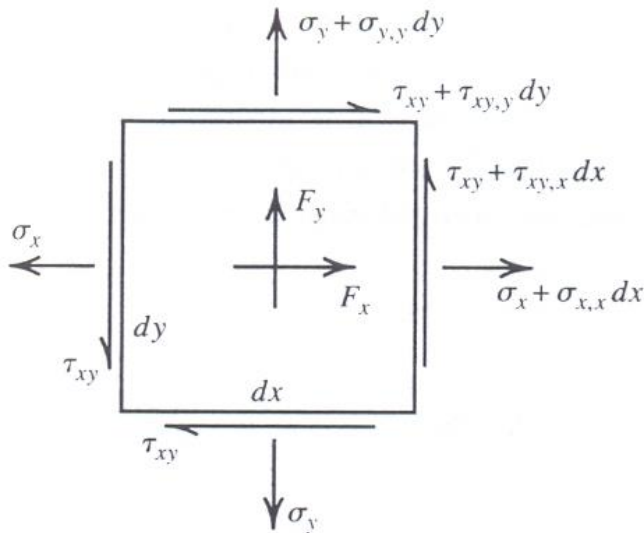
- Notice that **stress**  $\sigma_z$  may not be zero

$$\sigma_z = \nu(\sigma_x + \sigma_y) - E\alpha T$$





# Equations of equilibrium



$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

atau

$$\partial^T \sigma + b = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + b_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0$$

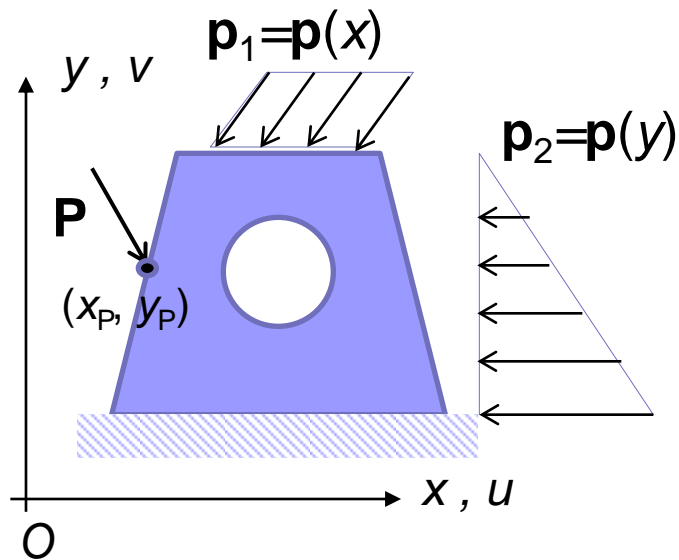
# Strong form problem statement

- Given geometrical and material properties and external actions  $\mathbf{P}$ ,  $\mathbf{p}$ ,  $\mathbf{b}$ ,  $T$ , and support displacement  $u_0$ , find  $\mathbf{u}(x,y)$  that satisfies:
  - Strain-displacement equations:  $\boldsymbol{\varepsilon} = \partial \mathbf{u}$
  - Stress-strain equations:  $\boldsymbol{\sigma} = \mathbf{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0)$
  - Equations of equilibrium:  $\partial^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$
- on the whole body and satisfies the given boundary conditions.

# The principle of virtual work

- The FEM does not directly use the strong form of governing equations, instead it uses **the weak form** of the equations.  
obtained from the principle of virtual work.
- The weak form can be obtained using:
  - ☐ The principle of stationary potential energy
  - ☐ The principle of virtual work

$$\delta U = \delta W$$



$S_1$ : the surface on which  $\mathbf{p}_1$  acts  
 $S_2$ : the surface on which  $\mathbf{p}_2$  acts

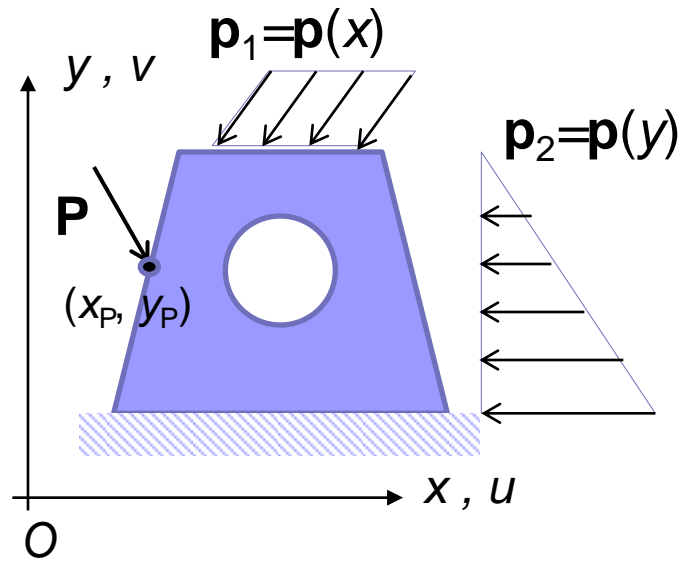
- The virtual strain energy for the body

$$\delta U = \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} dV$$

□  $\delta \boldsymbol{\epsilon}^T = \{\delta \epsilon_x \quad \delta \epsilon_y \quad \delta \gamma_{xy}\}$ , vector of virtual strains

- The external virtual work:

$$\begin{aligned}
 \delta W = & \delta \mathbf{u}(x_P, y_P)^T \mathbf{P} + \int_{S_1} \delta \mathbf{u}^T \mathbf{p}_1 dS \\
 & + \int_{S_2} \delta \mathbf{u}^T \mathbf{p}_2 dS \\
 & + \int_V \delta \mathbf{u}^T \mathbf{b} dV
 \end{aligned}$$



■ The principle of virtual work:

$$\begin{aligned}
 \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} dV \\
 = \delta \mathbf{u}(x_P, y_P)^T \mathbf{P} + \int_{S_1} \delta \mathbf{u}^T \mathbf{p}_1 dS + \int_{S_2} \delta \mathbf{u}^T \mathbf{p}_2 dS + \int_V \delta \mathbf{u}^T \mathbf{b} dV
 \end{aligned}$$

# Weak form problem statement

- Given geometrical and material properties and external actions  $\mathbf{P}$ ,  $\mathbf{p}$ ,  $\mathbf{b}$ ,  $T$ , and support displacement  $u_0$ , find  $\mathbf{u}(x,y)$  such that for all admissible  $\delta\mathbf{u}$

$$\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV = \delta\mathbf{u}^T \mathbf{P}(x_P, y_P) + \int_S \delta\mathbf{u}^T \mathbf{p} dS + \int_V \delta\mathbf{u}^T \mathbf{b} dV$$

- where  $\boldsymbol{\sigma}$  is defined in terms of  $\mathbf{u}$  using the strain-displacement and stress-strain relations



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- 3. Finite element formulation**
4. Isoparametric elements
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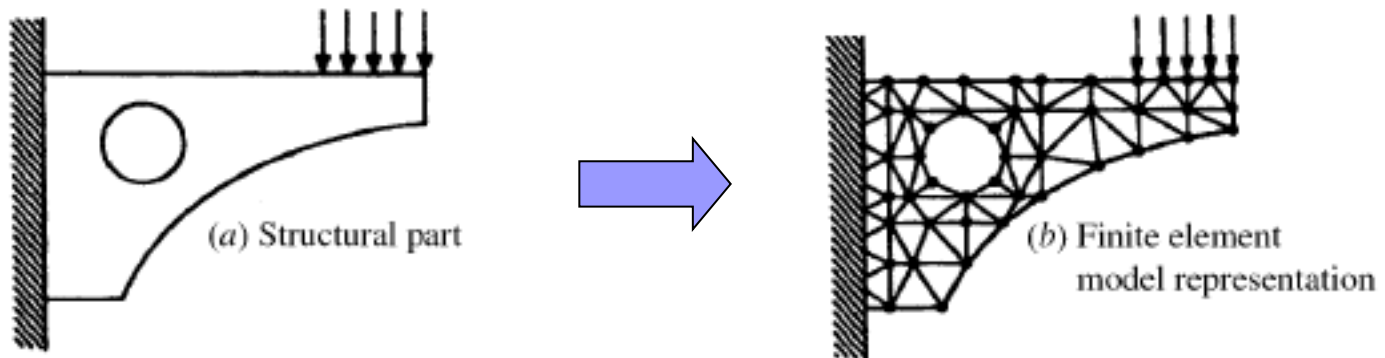
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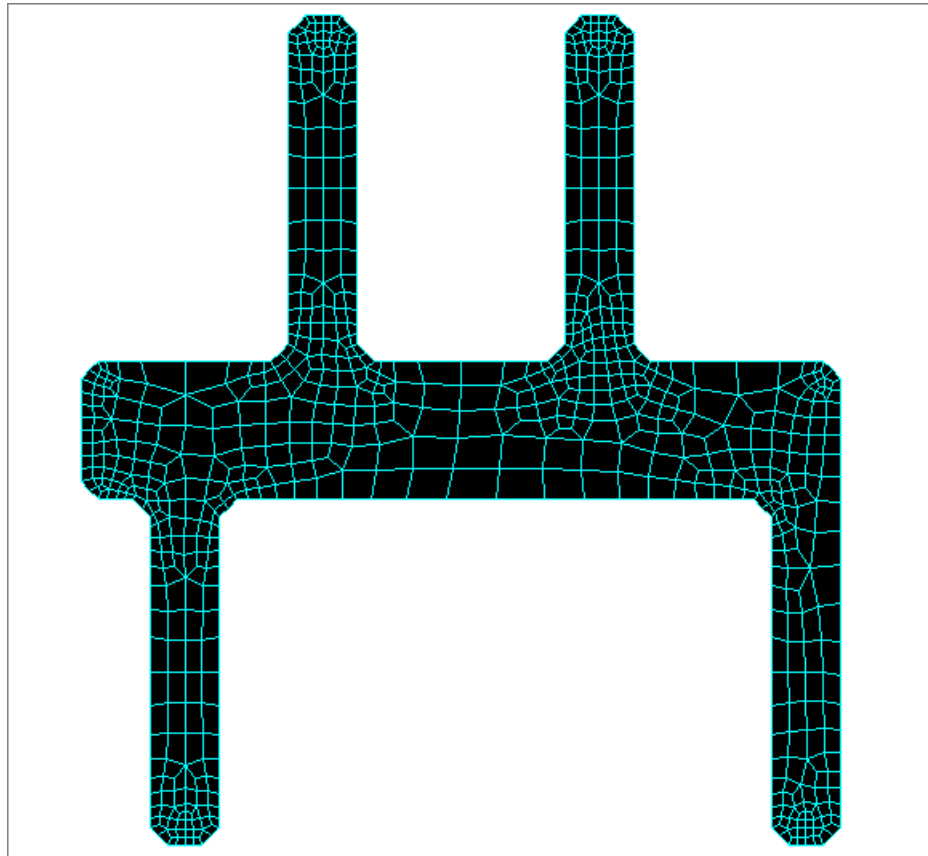


# Discretization (1)

- Fundamental concept is discretization, i.e. **dividing** a continuum (continuous body, structural system) into **a finite number of smaller and simple elements** whose union approximates the geometry of the continuum.



# Discretization (2)



Source: <http://members.ozemail.com.au/~comecau/autostep.htm>

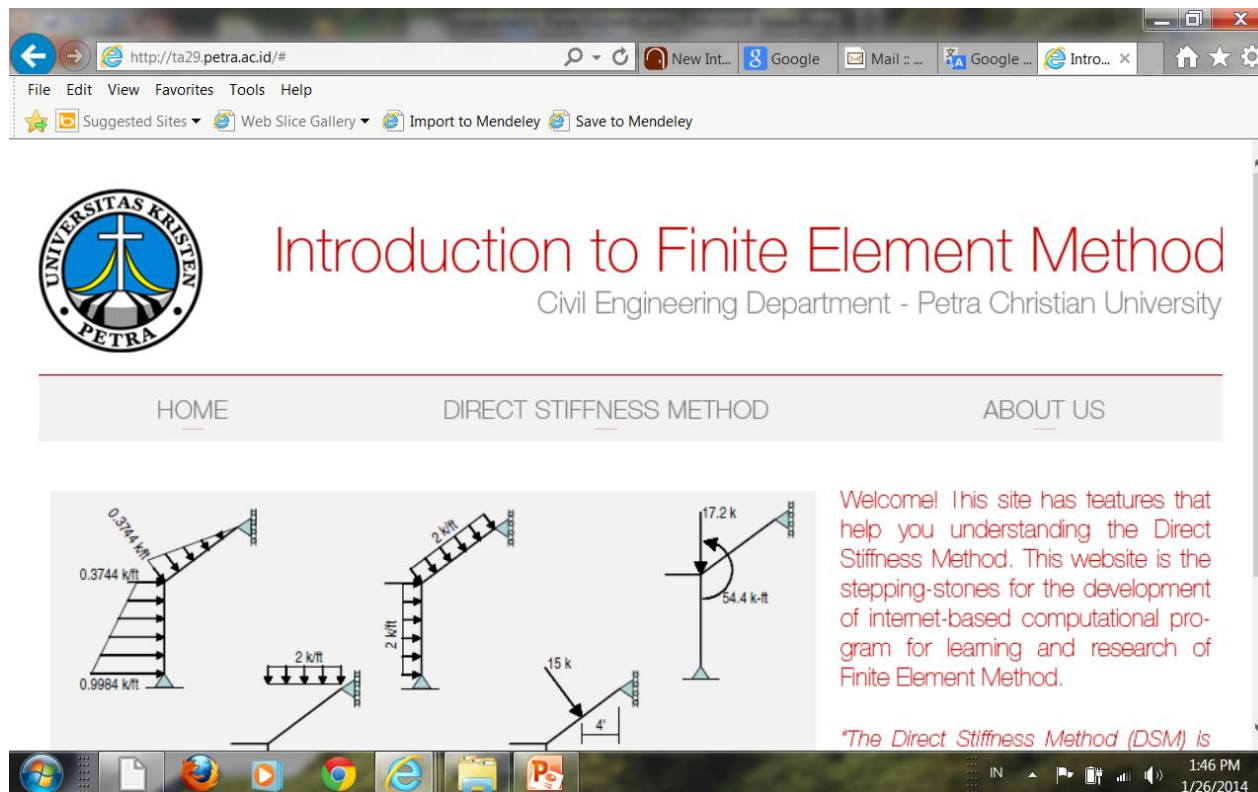
# Element formulation

- In FE formulation, we need to formulate an element to obtain the element stiffness equation

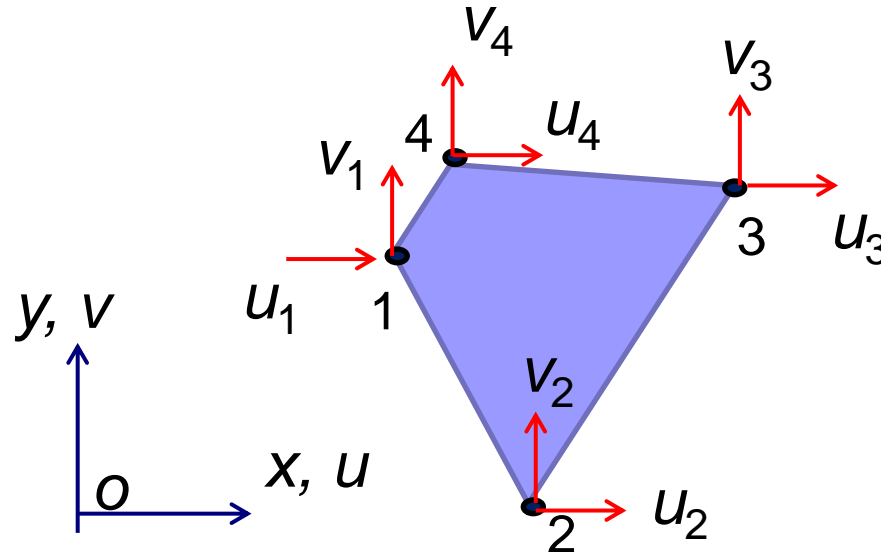
$$\mathbf{k}\mathbf{d} = \mathbf{f}$$

- Once we have this equation, the solution for the whole structure can be obtained using **the direct stiffness method**.

You may visit <http://ta29.petra.ac.id/#> to learn the direct stiffness method step-by-step (only for 2D frame structures)

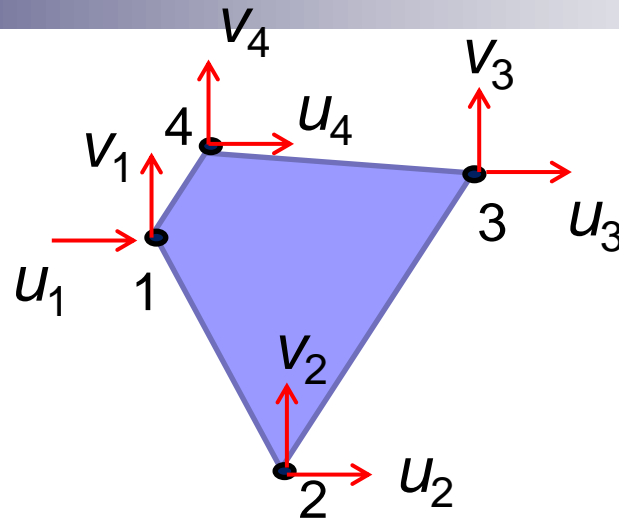


- Consider a quadrilateral element of the thickness  $h$  as illustrated here



- The displacement field within the element:

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix}$$



- Nodal displacement vector:

$$\mathbf{d} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4\}^T$$

- The first and fundamental step in the finite element formulation is to assume the displacement field within the element in terms of its nodal displacements.

- The assumed displacement field within the element can be expressed as:

$$u = \sum_{i=1}^4 N_i u_i$$
$$v = \sum_{i=1}^4 N_i v_i$$

- Or written in matrix form:

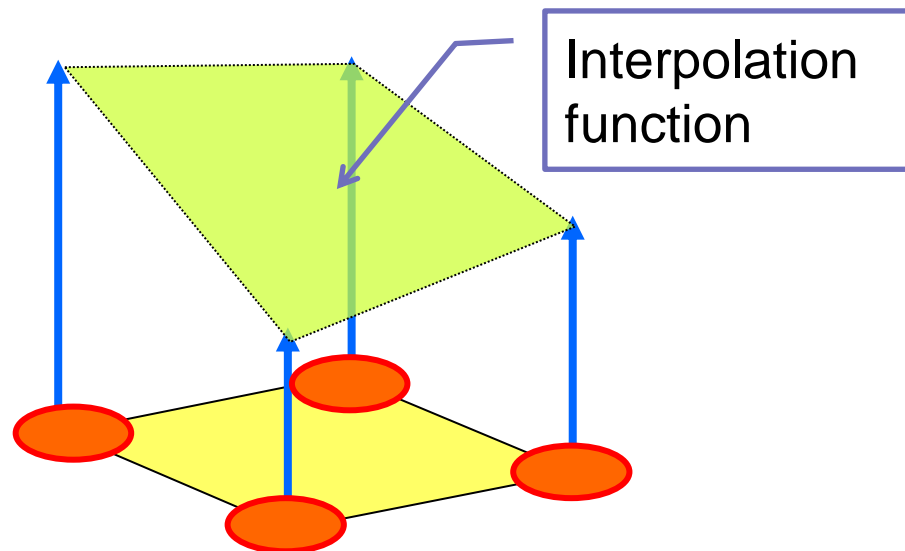
$$\mathbf{u} = \mathbf{N} \mathbf{d}$$

where

$$\mathbf{N} = \begin{Bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{Bmatrix}$$

$$\mathbf{u} = \mathbf{N}\mathbf{d}$$

- $\mathbf{N}$  is the matrix of shape functions
- $\mathbf{N}$  is also called the matrix of interpolation functions, because it interpolates the displacement field  $\mathbf{u}=\mathbf{u}(x, y)$  from the nodal displacements





- Strain-displacement relationships

$$\boldsymbol{\varepsilon} = \boldsymbol{\partial} \mathbf{u}$$

Matrix of differential operators

$$\boldsymbol{\varepsilon} = \boldsymbol{\partial} \mathbf{N} \mathbf{d}$$

- Thus we can write

$$\boldsymbol{\varepsilon} = \mathbf{B} \mathbf{d}$$

where

$$\mathbf{B} = \boldsymbol{\partial} \mathbf{N}$$

3x8

3x2

2x8

- Matrix  $\mathbf{B}$  gives strains at any point within the element due to unit values of nodal displacements.

- Stress-strain relationships (without considering temperature effect)

$$\sigma = E \varepsilon$$

thus

$$\sigma = E B d$$

- The principle of virtual work

$$\delta U_e = \delta W_e$$

where

$\delta U_e$  : the virtual strain energy of internal stresses

$\delta W_e$  : the virtual work of external forces on the elements

$$\delta U_e = \delta W_e$$

- Assume there exist a vector of small virtual displacements,  $\delta \mathbf{d}$
- The resulting virtual generic displacements and virtual strains are

$$\delta \underline{u} = \underline{N} \delta \underline{d}$$

$$\delta \underline{\varepsilon} = \underline{B} \delta \underline{d}$$

- The virtual strain energy of the element is

$$\delta U_e = \int_V \delta \underline{\varepsilon}^T \underline{\sigma} dV$$

$$\delta U_e = \int_V \delta \underline{\varepsilon}^T \underline{\sigma} dV$$

$$= \int_V (\underline{B} \delta \underline{d})^T (\underline{E} \underline{B} \underline{d}) dV$$

$$= \delta \underline{d}^T \left( \int_V \underline{B}^T \underline{E} \underline{B} dV \right) \underline{d} \quad (1)$$

- The external virtual work of nodal and body forces are

$$\delta W_e = \delta \underline{d}^T \underline{f} + \int_V \delta \underline{u}^T \underline{b} dV$$

$$= \delta \underline{d}^T \underline{f} + \int_V (\underline{N} \delta \underline{d})^T \underline{b} dV$$

$$= \delta \underline{d}^T \underline{f} + \delta \underline{d}^T \left( \int_V \underline{N}^T \underline{b} dV \right) \quad (2)$$

- Substituting the eqs. (1) and (2) into

$$\delta U_e = \delta W_e$$

and cancelling  $\delta \mathbf{d}^T$  from both sides of the eq. result in

$$\left( \int_V \underline{\mathbf{B}}^T \underline{\mathbf{E}} \underline{\mathbf{B}} dV \right) \underline{\mathbf{d}} = \underline{\mathbf{f}} + \int_V \underline{\mathbf{N}}^T \underline{\mathbf{b}} dV$$

- Thus, the element stiffness matrix is

$$\underline{\mathbf{k}} \equiv \int_V \underline{\mathbf{B}}^T \underline{\mathbf{E}} \underline{\mathbf{B}} dV$$

$$\left( \int_V \underline{B}^T \underline{E} \underline{B} dV \right) \underline{d} = \underline{f} + \int_V \underline{N}^T \underline{b} dV$$

- The equivalent nodal load vector due to body forces is

$$\underline{f}_b \equiv \int_V \underline{N}^T \underline{b} dV$$

- Let us rename the actual nodal force vector

$$\underline{f} \equiv \underline{f}_a$$

$$\underline{k} \underline{d} = \underline{f}$$

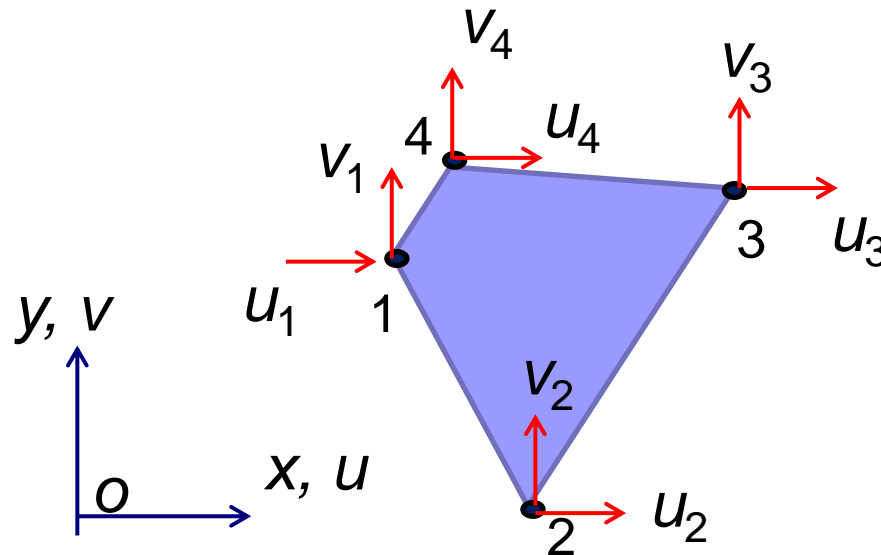
$$\underline{f} = \underline{f}_a + \underline{f}_b$$

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# Q4 Isoparametric Element

- Consider the quadrilateral element




- We need to obtain the expression for the shape functions  $N_1, \dots, N_4$

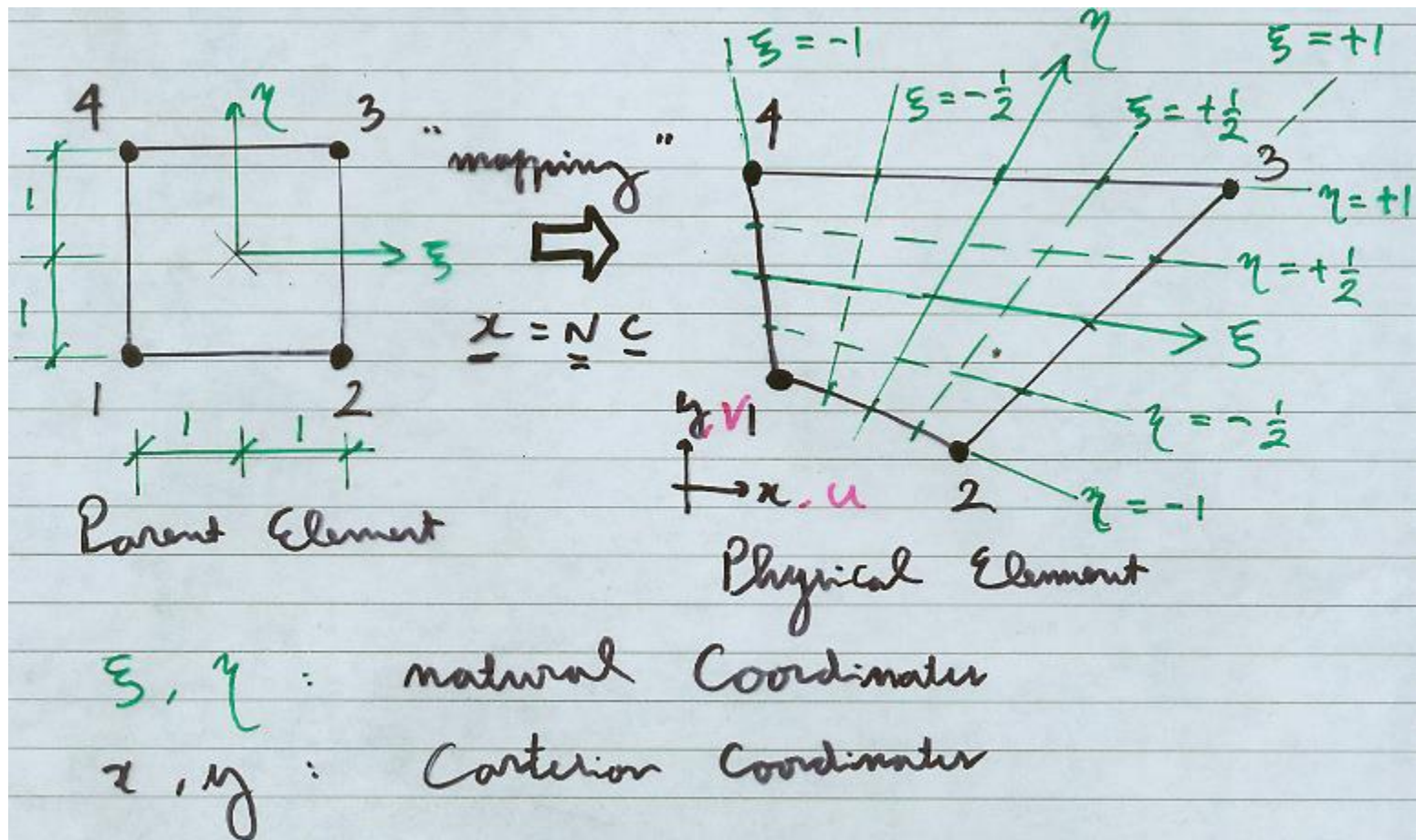


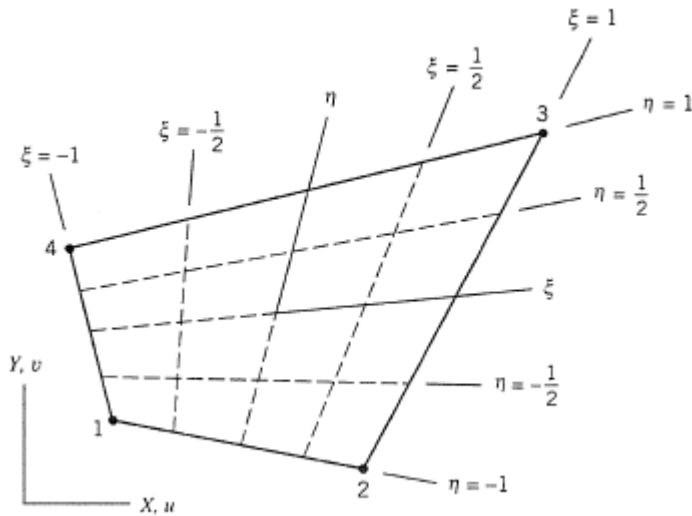
- 
- If we formulate the quadrilateral element directly in the cartesian coordinate system, we will face **technical difficulties**:
    - The expression for the shape functions is complicated
    - It is difficult to evaluate the integration in the stiffness matrix and equivalent nodal force vectors expressions exactly

- To overcome the difficulties and moreover, to facilitates the use of elements with curved edges, we map the element onto a square element defined in natural  $(\xi, \eta)$  coordinates.
  - The square element is called **parent** or **master element**.

- 
- To introduce the isoparametric concept, we will consider a type of quadrilateral element with four nodes for analysis of plane stress/plane strain problems.
  - This element is the standard “plane bilinear isoparametric element” (Q4).

# Mapping of the element





■ Nodal coordinates:

1  $(x_1, y_1)$

2  $(x_2, y_2)$

3  $(x_3, y_3)$

4  $(x_4, y_4)$

■ The geometry of the quadrilateral element is described by:

$$x = \sum_{i=1}^4 N_i x_i ; \quad y = \sum_{i=1}^4 N_i y_i$$

- Or, written in matrix form:

$$\mathbf{x} = \mathbf{N}\mathbf{c}$$

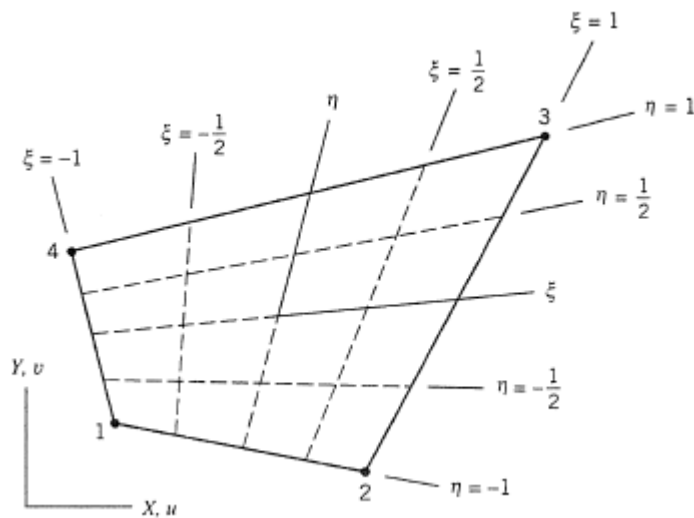
where

$$\mathbf{N} = \begin{Bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{Bmatrix}$$

and

$$\mathbf{c} = \{x_1 \quad y_1 \quad x_2 \quad y_2 \quad x_3 \quad y_3 \quad x_4 \quad y_4\}^T$$

- The shape functions are functions of natural coordinates  $\xi$  and  $\eta$ .



$$N_1 = \frac{1}{4} (1 - \xi)(1 - \eta)$$

$$N_2 = \frac{1}{4} (1 + \xi)(1 - \eta)$$

$$N_3 = \frac{1}{4} (1 + \xi)(1 + \eta)$$

$$N_4 = \frac{1}{4} (1 - \xi)(1 + \eta)$$

$(\xi, \eta) = (0, 0)$  is the center of the element


$$\mathbf{u} = \mathbf{N}\mathbf{d}$$

Displacement interpolation

$$\mathbf{x} = \mathbf{N}\mathbf{c}$$

Geometry interpolation

- The element is called bi-linear because the shape functions  $N_i$  are **linear** in  $\xi$  and **linear** in  $\eta$ .
- It is called isoparametric because the shape functions for interpolation of the geometry are **the same** as those for interpolation of the displacement field.



- The strain vector is

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_{,x} \\ u_{,y} \\ v_{,x} \\ v_{,y} \end{Bmatrix}$$

- The math difficulty here is that we have to differentiate  $u$  and  $v$  with respect to  $x$  and  $y$ , but  $u$  and  $v$  are functions of  $\xi$  and  $\eta$ .

- To overcome the difficulty, we apply the **chain rule** for partial differentiation

$$f = f(x, y)$$

$$\begin{aligned}x &= x(\xi, \eta) \\ y &= y(\xi, \eta)\end{aligned}$$

- Then,

$$\begin{aligned}\frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \xi} \\ \frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \eta}\end{aligned}$$

- Written in matrix form,

$$\begin{Bmatrix} f',_{\xi} \\ f',_{\eta} \end{Bmatrix} = \begin{bmatrix} x',_{\xi} & y',_{\xi} \\ x',_{\eta} & y',_{\eta} \end{bmatrix} \begin{Bmatrix} f',_x \\ f',_y \end{Bmatrix}$$

- The matrix  $\begin{bmatrix} x',_{\xi} & y',_{\xi} \\ x',_{\eta} & y',_{\eta} \end{bmatrix}$  is Jacobian matrix, **J**.

- Inverting the equation we obtain

$$\begin{Bmatrix} f',_x \\ f',_y \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} f',_{\xi} \\ f',_{\eta} \end{Bmatrix}$$

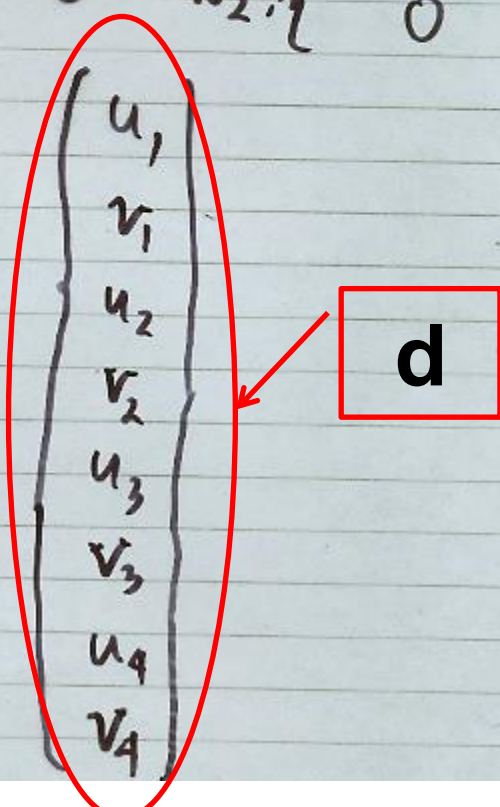
- Let define  $\Gamma$  as the inverse of the Jacobian matrix

$$J^{-1} = \Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$$

- Thus the derivatives of  $u$  and  $v$  can be written as

$$\begin{Bmatrix} u_x \\ u_y \\ v_x \\ v_y \end{Bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{11} & \Gamma_{12} \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{Bmatrix} u_\xi \\ u_\eta \\ v_\xi \\ v_\eta \end{Bmatrix}$$

$$\begin{Bmatrix} u_{1,3} \\ u_{1,7} \\ v_{1,3} \\ v_{1,7} \end{Bmatrix} = \begin{bmatrix} N_{1,3} & 0 & N_{2,3} & 0 & N_{3,3} & 0 \\ N_{1,7} & 0 & N_{2,7} & 0 & N_{3,7} & 0 \\ 0 & N_{1,3} & 0 & N_{2,3} & 0 & N_{3,3} \\ 0 & N_{1,7} & 0 & N_{2,7} & 0 & N_{3,7} \end{bmatrix} \begin{Bmatrix} N_{4,3} & 0 \\ N_{4,7} & 0 \\ 0 & N_{4,3} \\ 0 & N_{4,7} \end{Bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$



- Since  $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}$ , it can be concluded that the strain-displacement matrix is

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{11} & \Gamma_{12} \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \end{bmatrix} \chi$$

$$\begin{bmatrix} N_{1,\xi} & 0 & N_{2,\xi} & 0 & N_{3,\xi} & 0 & N_{4,\xi} & 0 \\ N_{1,\eta} & 0 & N_{2,\eta} & 0 & N_{3,\eta} & 0 & N_{4,\eta} & 0 \\ 0 & N_{1,\xi} & 0 & N_{2,\xi} & 0 & N_{3,\xi} & 0 & N_{4,\xi} \\ 0 & N_{1,\eta} & 0 & N_{2,\eta} & 0 & N_{3,\eta} & 0 & N_{4,\eta} \end{bmatrix}$$

## ■ The derivatives of the shape functions

$$N_{1,\xi} = -\frac{1}{4}(1-\eta) \quad ; \quad N_{1,\eta} = -\frac{1}{4}(1-\xi)$$

$$N_{2,\xi} = \frac{1}{4}(1-\eta) \quad ; \quad N_{2,\eta} = -\frac{1}{4}(1+\xi)$$

$$N_{3,\xi} = \frac{1}{4}(1+\eta) \quad ; \quad N_{3,\eta} = \frac{1}{4}(1+\xi)$$

$$N_{4,\xi} = -\frac{1}{4}(1+\eta) \quad ; \quad N_{4,\eta} = \frac{1}{4}(1-\xi)$$

## ■ Is **B** a polynomial function?



- The Jacobian matrix for the bilinear element can be written as

$$\begin{aligned}
 J &= \begin{bmatrix} x_{,5} & y_{,5} \\ x_{,7} & y_{,7} \end{bmatrix} = \begin{bmatrix} \sum N_{i,5} x_i & \sum N_{i,5} y_i \\ \sum N_{i,7} x_i & \sum N_{i,7} y_i \end{bmatrix} \\
 &= \begin{bmatrix} N_{1,5} & N_{2,5} & N_{3,5} & N_{4,5} \\ N_{1,7} & N_{2,7} & N_{3,7} & N_{4,7} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}
 \end{aligned}$$



- The element stiffness matrix is

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} dV = h \int_A \mathbf{B}^T \mathbf{E} \mathbf{B} dA$$

- The integration can be carried out in the isoparametric space, over the parent element, efficiently and accurately.
- To illustrate the integration, let consider the computation of the area of an element

## ■ The area of an element:

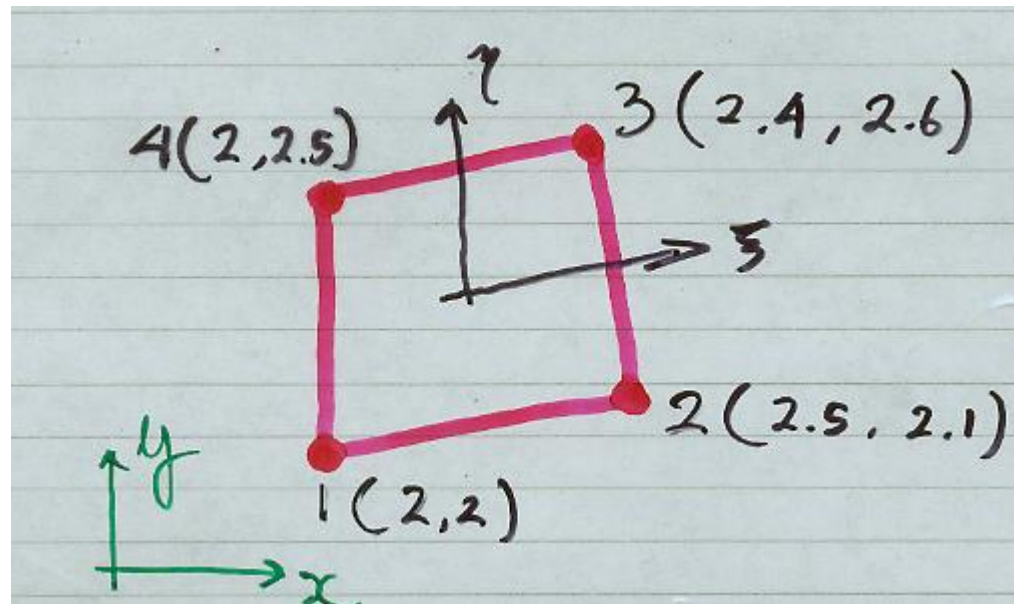
$$A_{\text{ren}} = \int_A dx dy = \int_A |J| d\xi d\eta$$

$\uparrow$  physical element       $\uparrow$  parent element

$|J|$ : determinant of jacobian matrix

## ■ Example:

Calculate the area of the element as shown in the following figure.



Hint:

$$Area = \int_A dA = \int_{-1}^1 \int_{-1}^1 \underbrace{|J|}_{0.23} d\xi d\eta$$

# Stiffness Matrix

$$\begin{aligned}\underline{\underline{k}} &= \iint \underline{\underline{B}}^T \underline{\underline{E}} \underline{\underline{B}} h \, dx \, dy \\ &= \int_{-1}^1 \int_{-1}^1 \underline{\underline{B}}^T \underline{\underline{E}} \underline{\underline{B}} h \, |\underline{\underline{J}}| \, d\xi \, d\eta\end{aligned}$$

$8 \times 8$        $8 \times 3$      $3 \times 3$      $3 \times 8$

# Equivalent Nodal Loads

Due to body force :

$$\underline{\underline{f}}_b = \int_{-1}^1 \int_{-1}^1 \underline{\underline{N}}^T \underline{\underline{B}} h \, |\underline{\underline{J}}| \, d\xi \, d\eta$$

## ■ Equivalent nodal loads

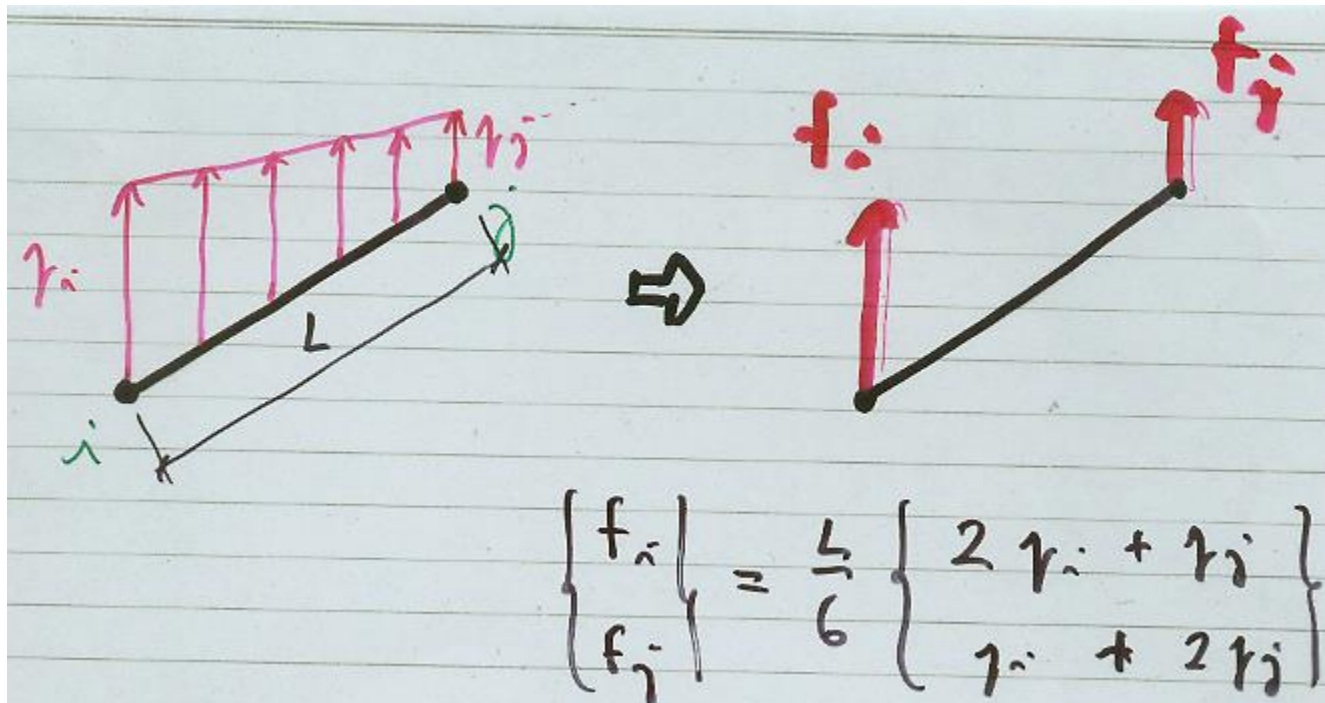
Due to initial strain  $\underline{\underline{\epsilon}}_0$  :

$$\underline{\underline{f}}_0 = \int_{-1}^1 \int_{-1}^1 \underline{\underline{B}}^T \underline{\underline{E}} \underline{\underline{\epsilon}}_0 h \left| \underline{\underline{J}} \right| d\xi d\eta$$

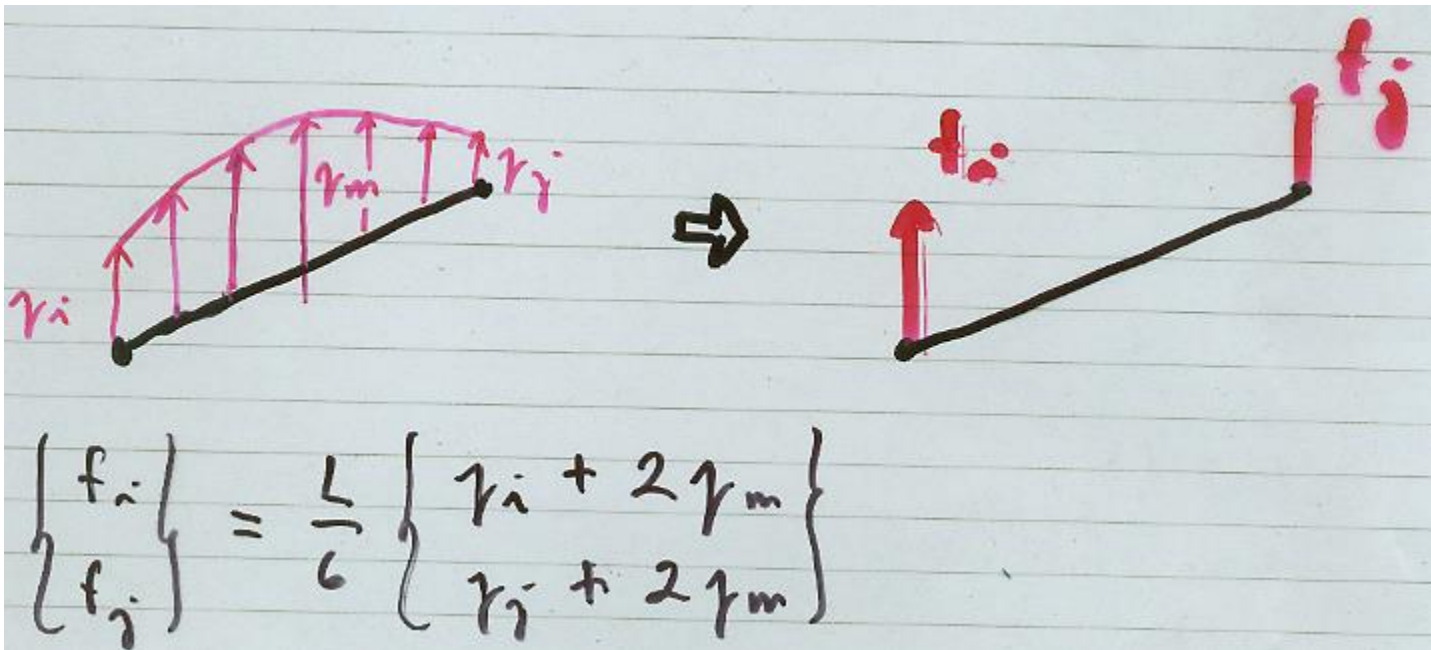
Due to traction forces :

$$\underline{\underline{f}}_t = \int_{A_{true}} \underline{\underline{N}}^T \underline{\underline{t}} dA = h \int_l \underline{\underline{N}}^T \underline{\underline{t}} dl$$

## ■ Practical formulas for evaluating equivalent nodal loads

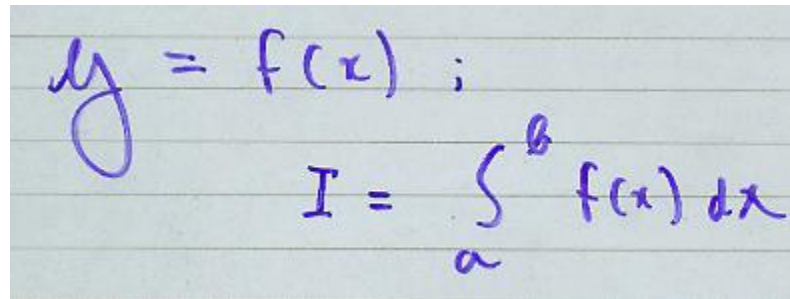






# Gauss Quadrature

- Quadrature: evaluating an integral numerically
- For 1D integration:


$$dy = f(x);$$
$$I = \int_a^b f(x) dx$$

- First, transform from arbitrary integration limits to  $-1$  to  $+1$



- The transformation (mapping) is given by

$$x = \frac{1}{2}(1-\xi)a + \frac{1}{2}(1+\xi)b$$

- Thus,

$$I = \int_a^b f(x) dx = \int_{-1}^1 \phi(\xi) d\xi$$

- Here  $\phi(\xi)$  incorporates the Jacobian of the transformation,

$$J = \frac{dx}{d\xi}$$

- Then evaluate the integral as follows,

A handwritten equation on lined paper showing the approximation of an integral. The equation is 
$$I = \int_{-1}^1 \phi(\xi) d\xi \approx \sum_{i=1}^{n_{\text{sample}}} w_i \phi(\xi_i)$$
. Below the equation, the text "Weighting factor" is written in green ink with an arrow pointing to  $w_i$ . To the right, the text "sampling point" is written in green ink with an arrow pointing to  $\xi_i$ .

- A  $n$ -point Gauss quadrature can integrate exactly a polynomial of degree  $2n-1$
- For a nonpolynomial function, the result will not be exact-- the higher the number of sampling points, the better the accuracy of the result.

ORDER  
n

$\xi_i$

$W_i$

1

0

2

2

$\pm \frac{1}{\sqrt{3}}$

1

3

$\pm \sqrt{0.6}$

$\frac{5}{9}$

0

$\frac{8}{9}$

4

$\pm \left[ \frac{3 + 2\sqrt{1.2}}{7} \right]^{\frac{1}{2}}$

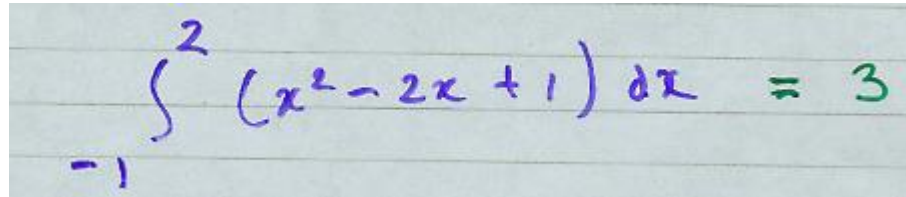
$\frac{1}{2} - \frac{1}{6\sqrt{1.2}}$

$\pm \left[ \frac{3 - 2\sqrt{1.2}}{7} \right]^{\frac{1}{2}}$

$\frac{1}{2} + \frac{1}{6\sqrt{1.2}}$

## □ Examples:

Using two sampling point, we can



A photograph of a handwritten equation on lined paper. The equation is  $\int_{-1}^2 (x^2 - 2x + 1) dx = 3$ . The integrand  $(x^2 - 2x + 1)$  and the limits  $-1$  and  $2$  are written in blue ink, while the equals sign and the result  $3$  are written in green ink.

This intergral cannot be exacly evaluated:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \exp\left(-\frac{1}{2}x^2\right) dx \approx 0.5$$

- For 2D integration:

$$\phi = \phi(\xi, \eta)$$

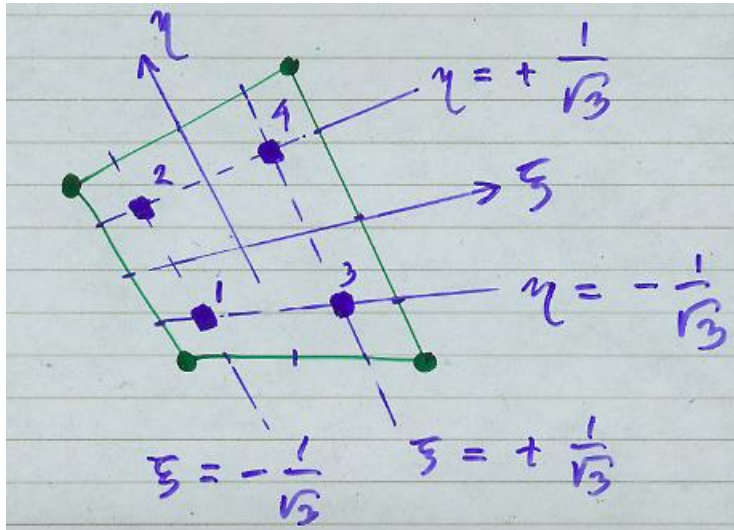
$$I = \int_{-1}^1 \int_{-1}^1 \phi(\xi, \eta) d\xi d\eta$$

$$\approx \int_{-1}^1 \left[ \sum_i w_i \phi(\xi_i, \eta) \right] d\eta$$

$$\approx \sum_j w_j \left[ \sum_i w_i \phi(\xi_i, \eta_j) \right]$$

$$= \sum_i \sum_j w_i w_j \phi(\xi_i, \eta_j)$$

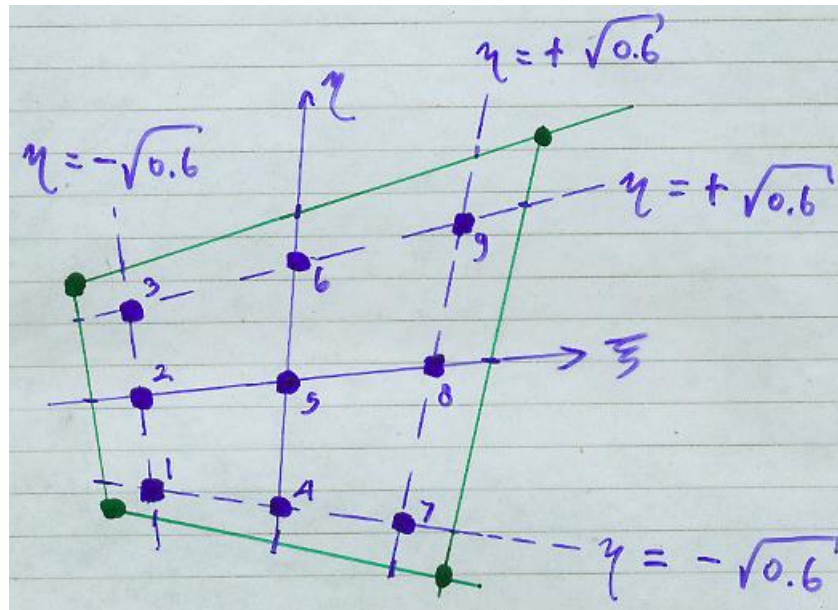
- Gauss point locations in a quadrilateral element using:
  - Four points (order 2 rule)



$$I \approx \phi_1 \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + \phi_2 \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + \phi_3 \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + \phi_4 \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$



## □ Nine points (order 3 rule)



$$I \approx \frac{25}{81} (\phi_1 + \phi_3 + \phi_7 + \phi_9) \\ + \frac{40}{81} (\phi_2 + \phi_4 + \phi_6 + \phi_8) + \frac{64}{81} \phi_5$$

# Computation of stiffness matrix

$$\underline{\underline{k}} = \int_{-1}^1 \int_{-1}^1 \underline{\underline{B}}^T \underline{\underline{E}} \underline{\underline{B}} h |\underline{\underline{J}}| d\xi d\eta$$

$$\approx h \sum_i \sum_j w_i w_j \left[ \underline{\underline{B}}^T \underline{\underline{E}} \underline{\underline{B}} |\underline{\underline{J}}| \right]_{(\xi_i, \eta_j)}$$



- If we use four sampling points:

$$\begin{aligned}
 \underline{\underline{k}} &= \underline{\underline{k}}_1 \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \\
 &+ \underline{\underline{k}}_2 \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\
 &+ \underline{\underline{k}}_3 \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \\
 &+ \underline{\underline{k}}_4 \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}
 \end{aligned}$$

$$\underline{\underline{k}} = \underline{\underline{k}}_1 + \underline{\underline{k}}_2 + \underline{\underline{k}}_3 + \underline{\underline{k}}_4$$

$\underline{\underline{k}}_i$ : stiffness matrix at sampling point  $i$

# Computation of equivalent body force

$$\begin{aligned}\underline{f}_b &= \int_{-1}^1 \int_{-1}^1 \underline{N}^T \underline{b} h |\underline{J}| d\xi d\eta \\ &= h \sum_i \sum_j w_i w_j \left[ \underline{N}^T \underline{b} |\underline{J}| \right]_{(\xi_i, \eta_j)}\end{aligned}$$

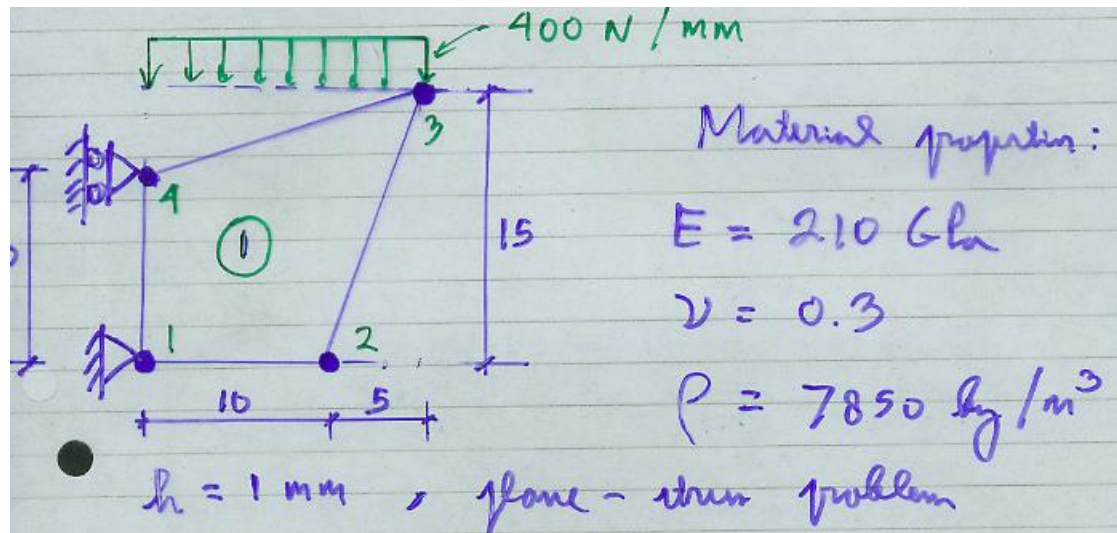
# Stress evaluation

- Once we have obtained the nodal displacement  $\mathbf{d}$  from the direct stiffness method, we can evaluate the stress at a point of interest within the element using

$$\underline{\sigma} = \underline{E} (\underline{B} \underline{d} - \underline{\epsilon}_0) + \underline{\sigma}_0$$

- We may evaluate the stress directly at the nodes.

# ■ Example



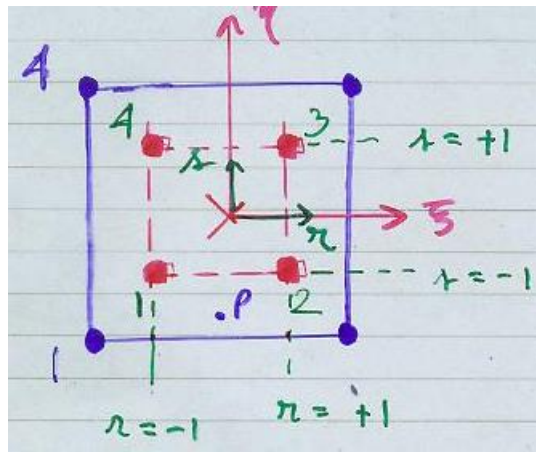
Node 1:  $\xi = -1, \eta = -1$

$$\underline{\sigma}_1 = \underline{E} \underline{B}(-1, -1) \underline{d}$$

Node 4:  $\xi = -1, \eta = +1$

$$\underline{\sigma}_4 = \underline{E} \underline{B}(-1, +1) \underline{d}$$

- However, according to many FEM texts, more accurate results can be achieved if we evaluate the stresses first at the Gauss points and then the stresses **at the nodes** are **extrapolated** from those at the Gauss points.



$$\eta = \xi \sqrt{3}$$

$$\xi = \eta \sqrt{3}$$

$$\sigma_P = \sum_{k=1}^4 N_k(\xi, \eta) \sigma_k$$

$$N_k = \frac{1}{4} (1 \pm \xi)(1 \pm \eta)$$

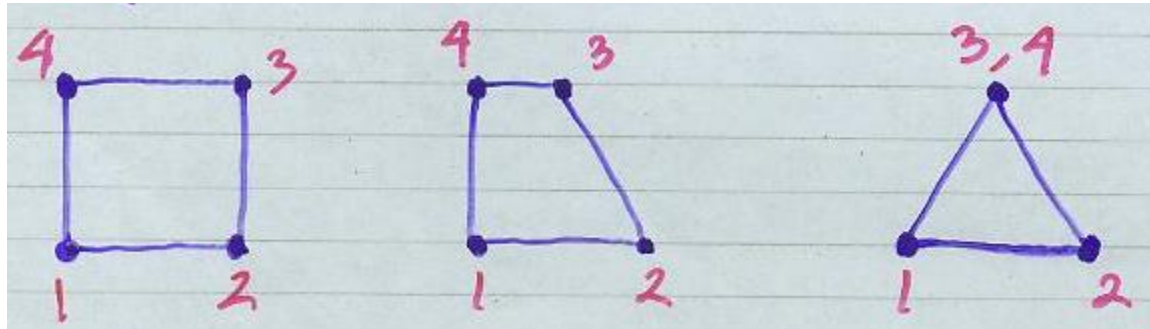
$N_k(\xi, \eta)$  are evaluated at the  $\xi$  and  $\eta$  coordinates of point P.

$$\text{Node 1: } \sigma_{(1)} = \sum_{k=1}^4 N_k(-\sqrt{3}, -\sqrt{3}) \sigma_k$$

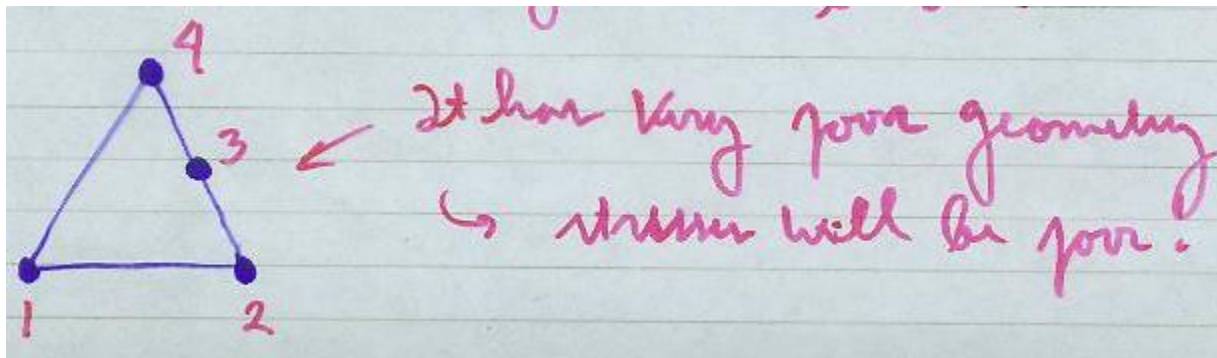
$$\sigma_{(4)} = \sum_{k=1}^4 N_k(-\sqrt{3}, \sqrt{3}) \sigma_k$$



# Degenerated Q4



- If nodes 3 and 4 are coincident, the quadrilateral becomes a triangle
- Another possibility



# Lecture Outline

1. Overview of the FEM
2. Governing equations of plane-strain/plane-stress problems
3. Finite element formulation
4. Isoparametric elements
5. **Element tests and applications**
6. References

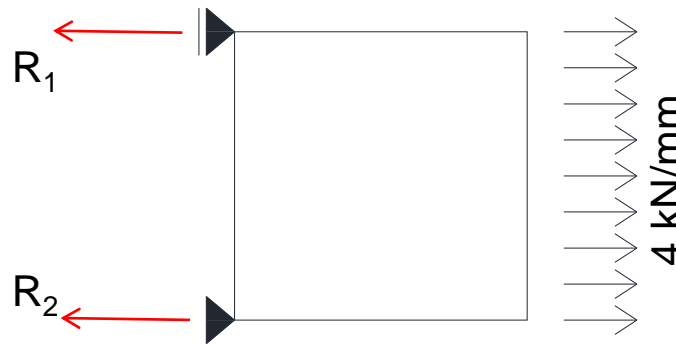


# 1. Pure tension problem

- For a thin plate in tension as shown in the figure, determine the nodal displacements, support reactions, and the stresses within the element



Dimension:  
400 mm x 400 mm  
 $h = 10$  mm  
 $E = 210$  kN/mm<sup>2</sup>  
 $\nu = 0.3$



- The displacement

$$\Delta = \frac{PL}{EA} = \frac{(4.400) \times 400}{210 \times (400.10)} = 0.761905 \text{ mm}$$

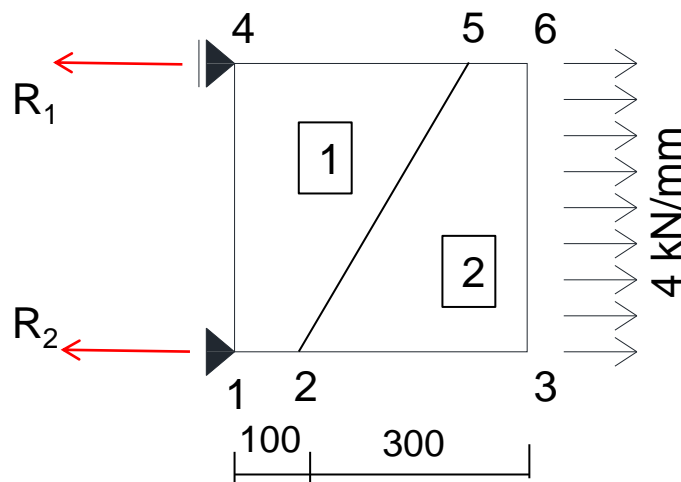
- The support reactions

$$R_1 = R_2 = \frac{400 \times 4}{2} = 800 \text{ kN}$$


- The stress within the whole body

$$\sigma = \frac{P}{A} = \frac{(4.400)}{(400.10)} = 0.4 \text{ kN/mm}^2$$

- These analytical results are useful for testing the performance of the Q4 element
- Suppose the the plate is discretized into two Q4 elements as follows

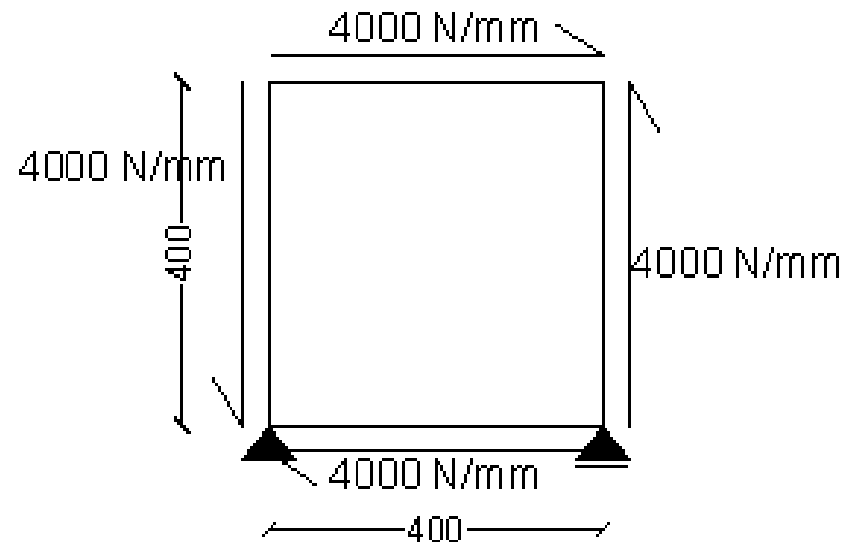


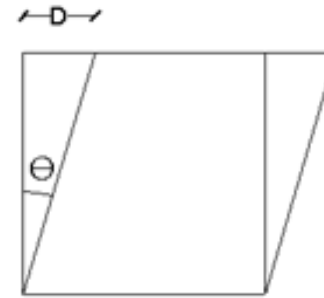
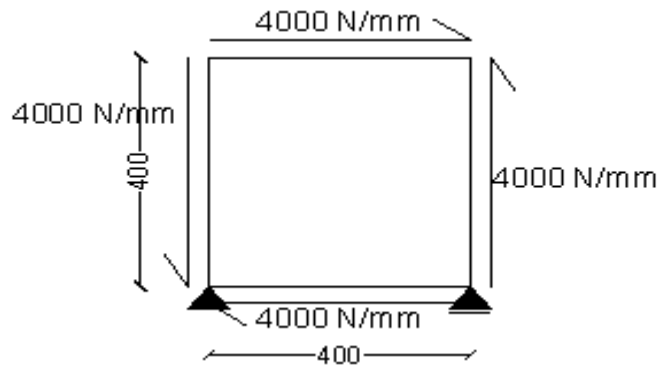
- Compare the results from the finite element to the exact solutions

- 
- See the Matlab files Pure\_tension\_dat.m then 'Go'
  - To see the resulting displacement, see the content of variable Xdisp, DOF number 5 and 11
  - To see the resulting stresses at the nodes, see the content of variables Stress
  - It can be seen that the resulting displacement, support reactions and stresses are all exact!
  - Please check using your commercial software, **Strand**.

## 2. Pure shear problem

- For the same thin plate but in pure shear stress state as shown in the figure, test the CST performance





- The shear stress and shear modulus:

$$\tau = \frac{p}{h} = \frac{4000}{10} = 400 \text{ N/mm}^2 = 0.4 \text{ kN/mm}^2$$

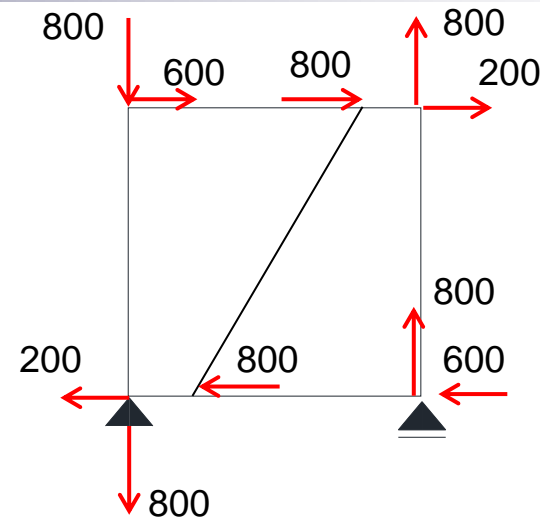
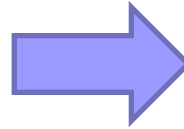
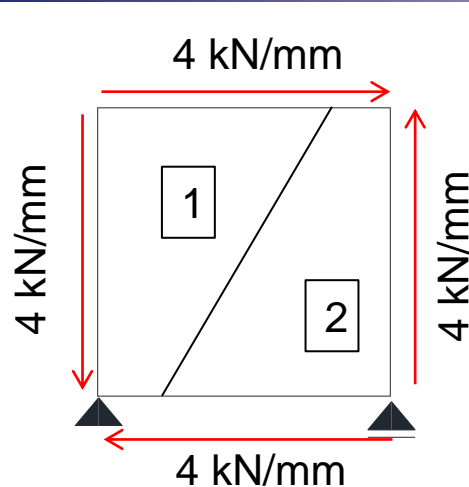
$$G = \frac{E}{2+2\nu} = \frac{210000}{2 + 2 \times 0.3} = 80769 \text{ N/mm}^2 = 80.769 \text{ kN/mm}^2$$

- The shear strain within the whole body

$$\gamma = \frac{\tau}{G} = \frac{400}{80769.23077} = 0.004952381$$

- The displacement of the top surface

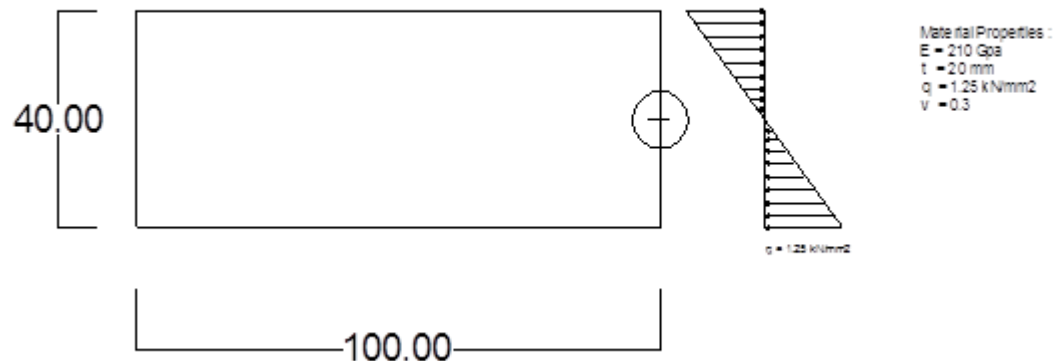
$$D = 400 \times 0.004952381 = 1.981 \text{ mm}$$



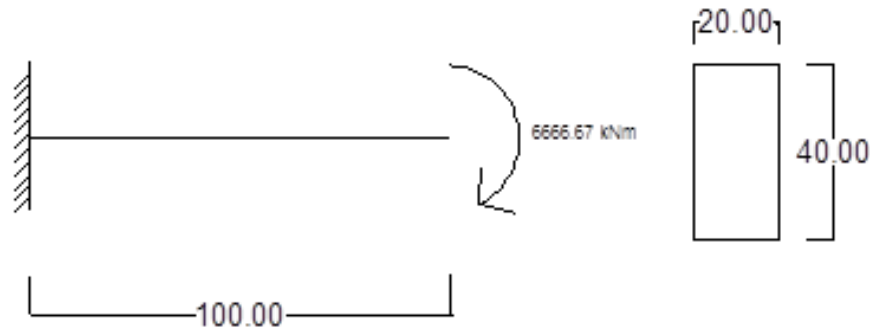
- See the Matlab files Pure\_shear\_dat.m then 'Go'
- See the content of variable Xdisp, DOF number 7, 9, and 11
- It can be seen that the resulting displacement, support reactions and stresses are all exact!
- How about the results from your software?

### 3. Pure bending problem

- Consider a two dimensional body under pure bending condition as shown in the figure.
- Test the performance of the Q4 in this problem.





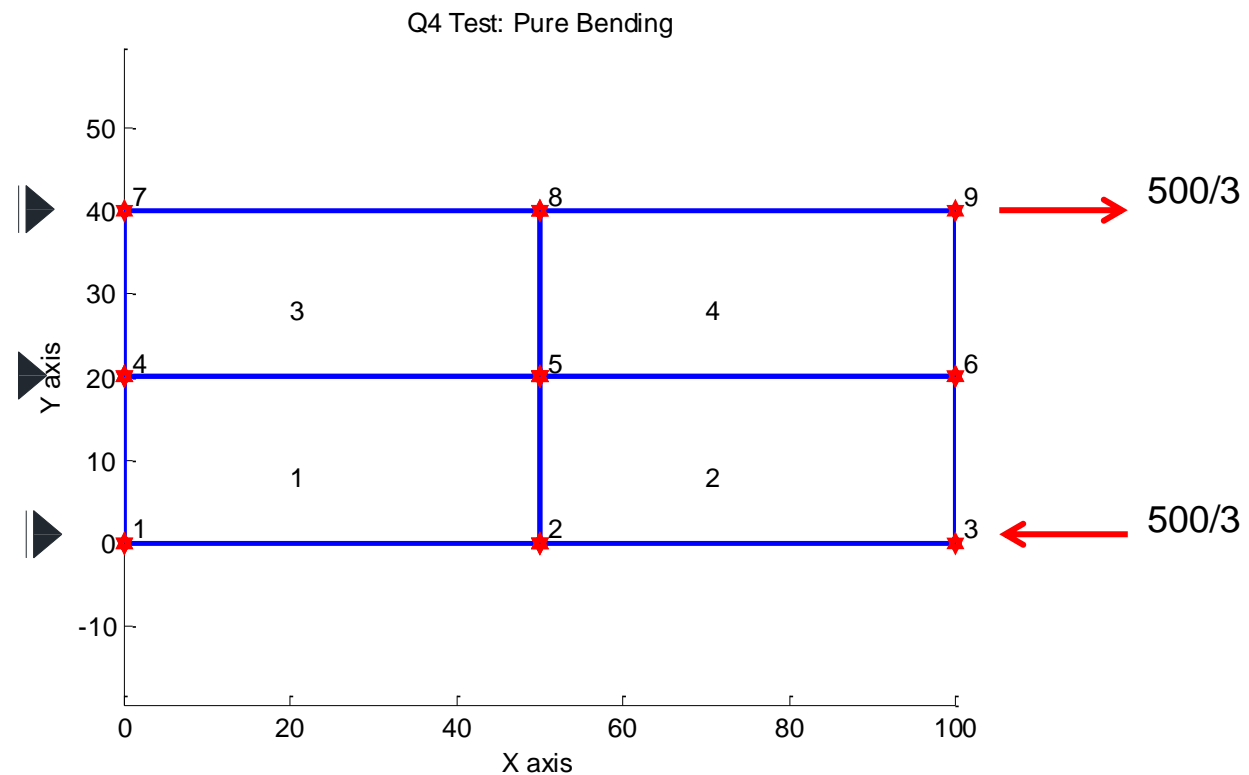


- The deflection of the neutral axis according to the beam bending theory:

$$\Delta = \frac{ML^2}{2EI} = \frac{6666.7 \times 100^2}{2 \times 210 \times \frac{1}{12} \times 20 \times 40^3}$$
$$= 1.4881 \text{ mm}$$

# ■ Finite element model

□ See Pure\_bending\_dat

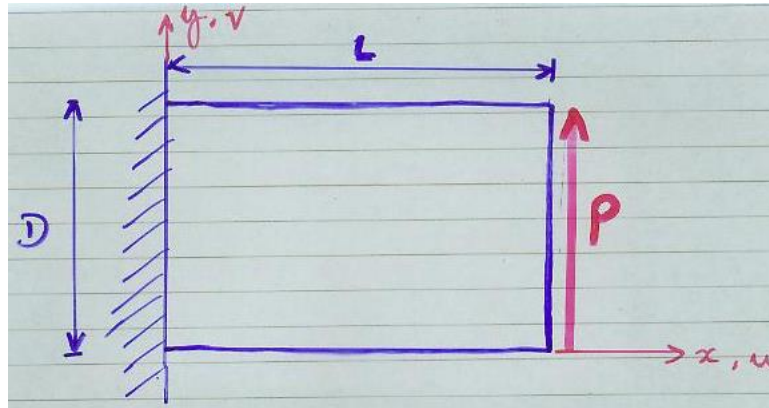


- The deflection of the neutral axis from FEA using different degrees of mesh refinement:

MESH	Q4	Percentage
2x2	0.9154	61.5%
4x4		
8x8		
16x16		
Exact	1.4881	100.0%

- It needs a fine mesh to obtain Q4 accurate solution.
- We can conclude that the performance of the Q4 is not so satisfactory in bending problem.

## 4. Plane Elasticity Beam

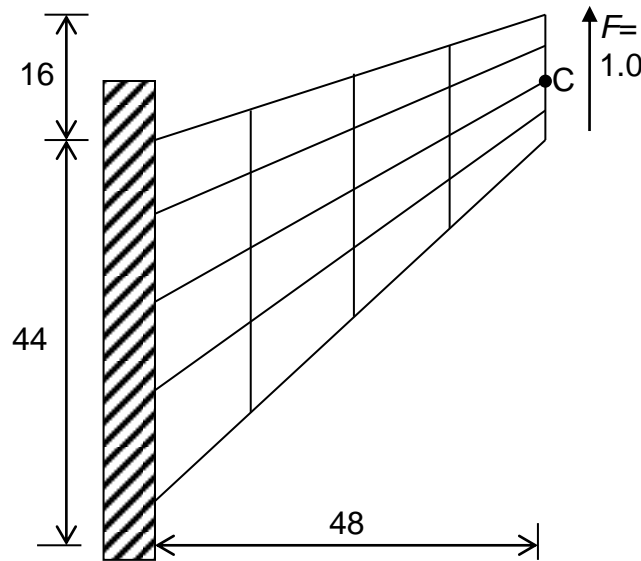


- The analytical solution for the (plane stress) cantilever beam problem is given by Timoshenko and Goodier as follows:

$$u = -\frac{P}{6EI} \left( y - \frac{D}{2} \right) \left[ 3x(2L-x) + (2+\nu) y(y-D) \right]$$

$$v = \frac{P}{6EI} \left[ x^2(3L-x) + 3\nu(L-x) \left( y - \frac{D}{2} \right)^2 + \frac{4+5\nu}{4} D^2 x \right]$$

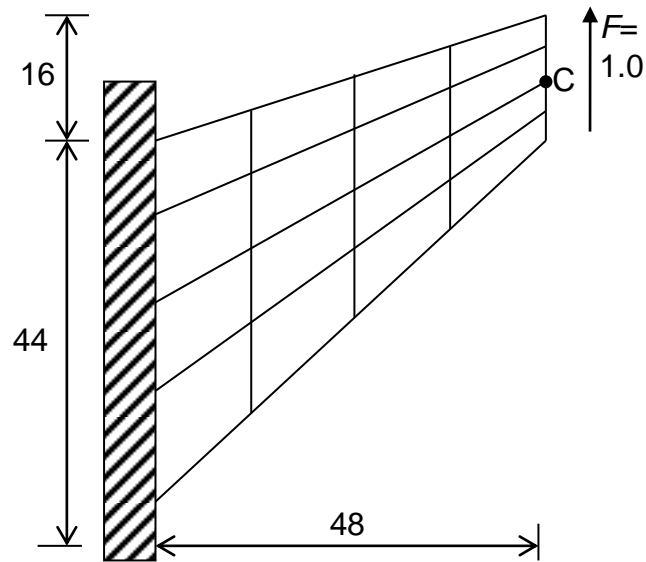
# 5. Cook's Membrane Problem



See  
Cook\_m4auto\_d  
at and  
Cook\_m8\_dat

$E=1.0$ ,  $\nu=1/3$ , and  $h=1.0$

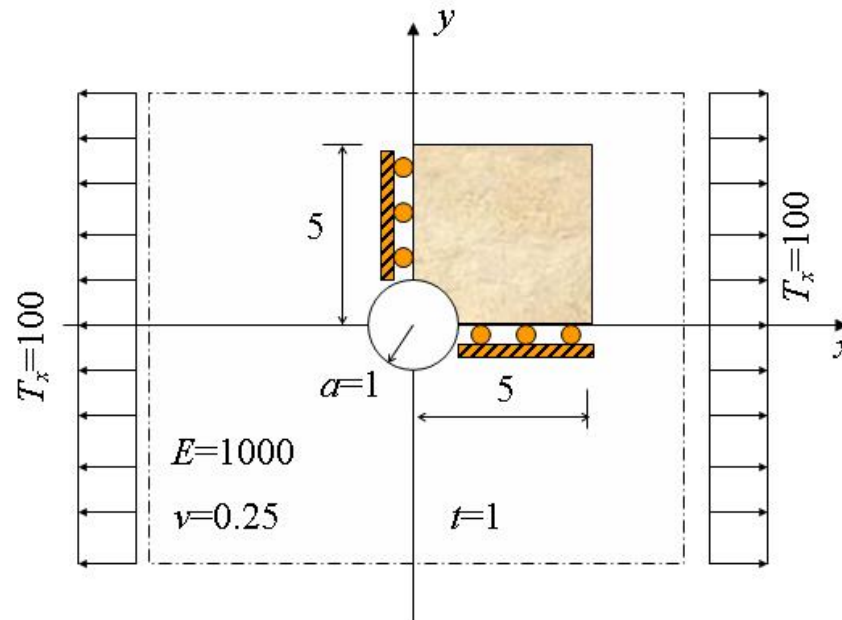
The reference solution:  $v_C=23.91$



# elements on each side	Q4	Percentage
4	18.30	76.5
8	22.08	92.3
16		
Ref. soluion	<b>23.91</b>	<b>100%</b>

## 6. An infinite plane-stress plate with a hole

The plate subjected to uniform tension  $T_x$  at infinity



# Lecture Outline

1. Overview of the FEM
2. Governing equations of plane-strain/plane-stress problems
3. Isoparametric elements
  - Bilinear isoparametric quadrilateral element
4. Element tests and applications
5. References



# References- Basics of the FEM

- D.L. Logan (2007)  
*A First Course in the Finite Element Method*  
the 4<sup>th</sup> Ed., Toronto, Nelson
- D.V. Hutton (2004)  
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New York, McGraw-Hill
- S.W. Alisjahbana (1998)  
*Prinsip Dasar Metode Elemen Hingga*  
Jakarta, UPT Penerbitan Universitas Tarumanagara
- P.B. Kosasih (2012)  
*Teori dan Aplikasi Metode Elemen Hingga*  
Yogyakarta, Andi Offset

# References- More Advance

- R. D. Cook, D.S. Malkus, M.E. Plesha and R.J. Witt (2002)  
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